THE
MATHEMATICAL GAZETTE

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THE USE OF A MIRROR IN THE TEACHING OF MATHEMATICS.*

BY C. W. HANSEL, BEDFORD SCHOOL.

1. INTRODUCTORY.

THE author of this paper has, for many years, used a plane mirror at 45 degrees with the table, to demonstrate the properties of rays and beams of light. A horizontal projection of rays must be viewed from above in the absence of a mirror. If a mirror is used, the horizontal layout is seen in a vertical plane in the mirror.

It is only recently, as the result of an A.R.P. demonstration on incendiary bombs, that the effectiveness of the mirror method of demonstration has become apparent as well as the frequent and diverse occasions in which it is useful. In the demonstration referred to, some houses, farm buildings, hay stacks, and the like were exhibited as models on a table. The point of striking of an incendiary bomb was registered by the lighting up of a small flash lamp. Scarcely anyone in the large audience could see what was happening. A mirror at 45 degrees would have rendered the whole proceedings visible to everyone.

Similarly, there are many cases in the teaching of mathematics when it is either inconvenient or impossible to demonstrate effectively on a vertical blackboard or on a vertical layout. The present demonstration will deal with examples of this kind and with some problems of applied mathematics which are effectively demonstrated with a mirror.

2. SIMPLE MATHEMATICS FOR BEGINNERS.

Demonstration 1.

1. The " weight " box.
2. Mensuration.
3. Pythagoras.
4. Ptolemy's theorem.
5. The vernier.

*An account of a paper and demonstration to the General Meeting of the Mathematical Association, April 13, 1944.

3. Measurement.

In recent years, a majority of the Science Masters' Association and the Board of Education have, by their preference for the teaching of general science, indicated an indifference to individual experimental work and quantitative discussion which seems to me in direct conflict with scientific ideals and scientific method. An analysis of the range of subject-matter covered and the time allocated to it by the syllabus of general science makes it quite clear that what it provides is instruction in the facts of science and not a scientific education—book-learning rather than education through experience. Would it not be better to persuade the historian to abandon political history as meat for school children and to substitute the history of ideas, of science, mathematics, and industry and the arts? In short, to teach the history of human progress instead of the history of inhuman strife and devilish cunning. Such historical teaching could be a substitute for general science, but merely supplementary to a training in scientific technique and in mathematics. Mathematics could profitably include all subjects dealing with measurement and quantitative problems. Mathematics, physics and engineering could easily be combined up to the standard of a pass degree.

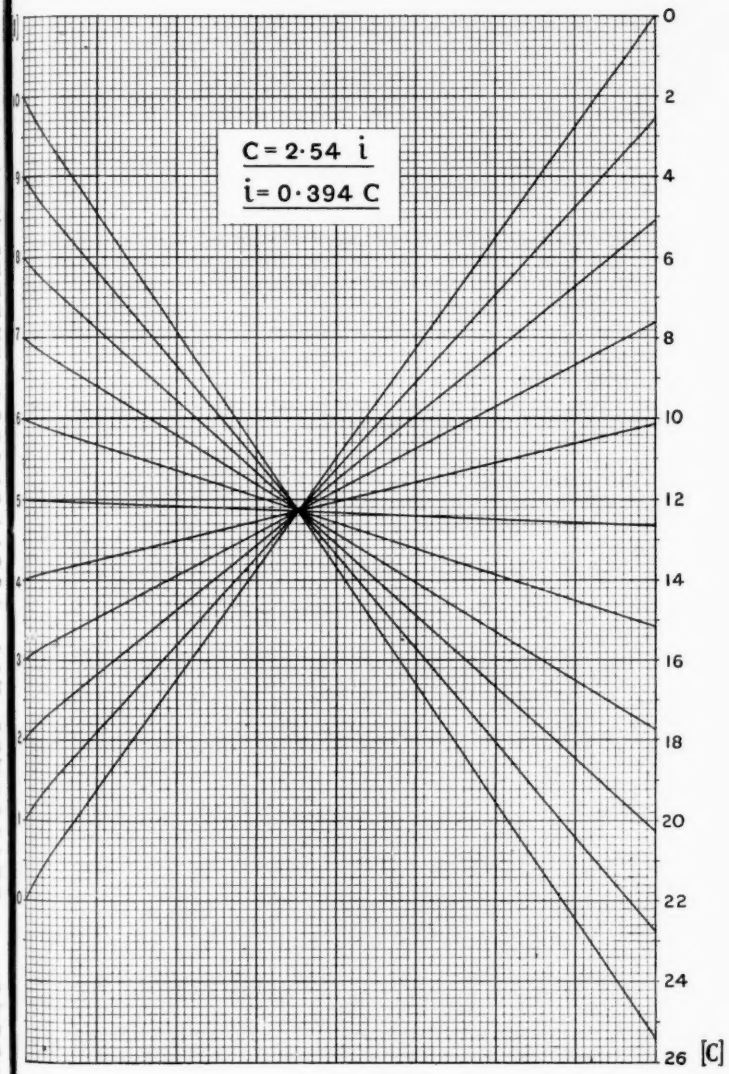
An Example.—Let a student put two rulers side by side, the centimetre scale of one opposite the inch scale of the other. Let him read off, with the greatest possible care, the number of centimetres (c) corresponding to various numbers of inches (i); for example, 1″, 2″, 3″, up to 10″. At a later stage the experiment may be repeated as an exercise on the vernier. Tabulate i, c, c/i, i/c. Discard obviously incorrect measurements and remeasure. Discuss the measurements and results, Avoid spurious accuracy. Plot c against i. Use the graph for interpolation and extrapolation. Discuss direct proportion. Show that $c = m \,.\, i$ where $m = \Delta c/\Delta i$. It is not necessary to represent more than 10 inches on the graph in order to deal with any number of inches. Use of scale factors. This introduces standard form, the index notation, movement of the decimal point.

A few comprehensive exercises of this kind combining experimental work, computation, graphical work, consideration of functionality and errors, units, etc., provide a type of training far superior to that derived from the working of isolated and unconnected examples.

The value of experimental work is greatly increased if the student strives persistently for the highest accuracy attainable and strains after the most desirable technique. Persistent conscious effort and critical discussion of technique bring a rich harvest. A careful description of each experiment should be given, illustrated, if possible, by carefully drawn diagrams. Here again is an opportunity for mental discipline of a high order and for practice in English composition, précis writing, the writing of a report and for practice in the communication of ideas by writing or sketch. Clear, concise, logical description involves critical and careful mental analysis and promotes thorough understanding and orderly thinking.

4. Nomography.

Formal instruction in nomography is not recommended, but nomograms may be introduced as and when it seems natural and useful to do so. For example: the figure relates to the experiment just described. [I] reads "scale I" and [C] reads "scale C". i, c denote corresponding readings on [I] and [C]. Corresponding scale readings of the measurements are connected by a straight line or "secor". The functional operation of drawing the secor is indicated symbolically by $i : c$, the colon being the symbol of operation. The secor $(i = 0) : (c = 0)$ is obviously independent of errors of


C = 2·54 i
i = 0·394 C
0
2
4
6
8
10
12
14
16
18
20
22
24
26
[C]

measurement. A child, at the age of ten, cannot fail to notice that the secors pass very nearly through a point on the secor $(i = 0) : (c = 0)$. If any secor is seriously out, remeasurement will reveal an error of measurement. The "best" point of intersection of the secors may be found by visual judgment. This simple method of representing experimental results and finding the best point is not inferior to the method of least squares and merely requires visual judgment as a criterion of rejection.

A similar nomogram may be used for converting other units, for example, cubic inches and cubic centimetres, temperatures, etc. It is not necessary to make measurements to construct the nomogram if the relation between the units is known. The same type of nomogram may be used to indicate direct proportionality or for the discovery of a linear law. It will also be apparent that with a suitable scale or scales along the line connecting the zeros of the vertical scales, the nomogram enables multiplication and division to be carried out.

Demonstration 2.

1. Conversion of units and direct proportion.
2. Linear law.
3. Addition and subtraction.
4. Multiplication and division.
5. Involution and evolution.
6. Flow of water through a pipe.
7. Reciprocal nomogram.
8. Square scales.
9. Quadratic equations.
10. $\delta y/\delta x$.
11. Curve plotting, using logarithmic and semi-logarithmic paper.

4. Statics and Dynamics.

4·1. Experimental Statics.

Two clamps of special design will support a drawing board or framework of laths, etc., and thus provide a universal method of mounting statics apparatus. Light springs may be used to measure forces with the aid of an antiparallax scale. The same scale may be used for any spring, and it may be graduated in pounds weight, poundals, grams weight, kilodynes. A peg or bolt attached to a large bulldog clip provides an easily movable and adjustable point of support.

Demonstration 3.

1. Triangle of forces.
2. Components of a force.
3. Graphical statics.

4·2. Experimental Dynamics.

The chief difficulty in experimental work in dynamics is timing the moving body. In the apparatus shown, the moving body is a steel ball which marks out its own time and distance with the aid of sensitive pen copying paper. The groove in the wooden track has a radius such that a steel ball of diameter 3·000 centimetres has a period of oscillation in the groove of 0·400 seconds. It is possible to measure an acceleration to an accuracy of 1 per cent. after taking differences twice between the readings of time and distance.

It is important to realise that the oscillating ball descends the inclined track with the same uniform acceleration as if it rolled straight down without oscillating. This provides an experimental illustration of the Galilean principle of the independence of perpendicular motions.

Demonstration 4.

1. Uniform acceleration in a straight line.
2. The parabolic trajectory of a projectile.

5. Optics.

According to Fermat's principle of stationary path, the time taken by a ray of light in passing from one point to another is, in general, a minimum. The three fundamental laws of light may be deduced from Fermat's principle.

Problems 1 and 2 are equivalent.

1. P and Q are points external to a straight line AB. Find a point R in AB so that PR and QR make equal angles with AB. Show that $PR+RQ$ has a minimum value.

2. Light from a point P is reflected at R by a plane mirror AB to Q. If light travels in straight lines and the angle of incidence is equal to the angle of reflection, show that $PR+RQ$ has a minimum value.

It is not the purpose of the present paper to dwell on the theorems of geometrical optics but the following may be referred to :

For an ellipsoidal mirror, all rays of light emitted from one focus P are reflected to the other focus Q ; P and Q are aplanatic points.

For a paraboloidal mirror, the foci are aplanatic points if P is the near focus, the focus Q is at infinity, and a beam of light diverging from P is reflected as a parallel beam of light.

Thomas Young's construction for the aplanatic points of a sphere of refractive index n (attributed to Weierstrass in Drude's *Optics*) shows that they are in pairs along any radius of the sphere ($=a$) and at distances a/n and an from the centre.

Demonstration 5.

The caustic curve for a hemi-cylindrical mirror.

The demonstration indicates how the reflected rays envelop a curve, the caustic curve. The reflected rays may be removed or returned one by one in order to show how the caustic curve is generated.

If a parallel beam of light is reflected by a hemi-cylindrical mirror of radius a, the caustic curve is one-half of a 2-cusped epicycloid generated by rolling a circle of radius $a/4$ on a circle of radius $a/2$ concentric with the hemi-cylinder. The cusp is at the centre of the radius which happens to be parallel to the rays of the beam.

Demonstration 6. Curve Tracing.

For this purpose, pen copying paper may be used (Zanetic paper). Bars, discs, wheels, etc., are drilled at suitable points so that the head of a protruding pin may be used as a tracing point which presses the recording paper to the copying paper.

The cycloid, epicycloid, involute and evolute curves may be traced.

Alternatively, mechanisms may be used on a horizontal board and the reflection in a vertical plane viewed by the audience. C. W. H.

WANTED.—A graduate with a knowledge of Mathematics, Physics, French and German, to act as Librarian. Must be able to develop a special interest in mathematical tables. Special consideration will be given to a partly disabled service man *with the necessary qualifications.*

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INFINITE SERIES FOR FIFTH-FORMERS.*

By N. M. Gibbins.

1. Attention was called to the graphical representation of a geometric progression by W. J. Dobbs, *Gazette*, IX, p. 245, as in Fig. 1. Such illustrations,

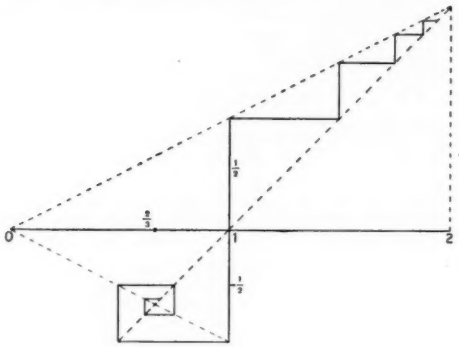


Fig. 1.

together with numerical examples, are sufficient for School Certificate examination purposes. The following considerations are intended to fill up time after the examination.

2. Other series may be introduced for term-by-term comparison with

$$1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\dots;$$

e.g., the series for e, and

$$1+\left(\frac{1}{2^2}+\frac{1}{3^2}\right)+\left(\frac{1}{4^2}+\dots+\frac{1}{7^2}\right)+\dots.$$

By way of contrast the series

$$1+\tfrac{1}{2}+(\tfrac{1}{3}+\tfrac{1}{4})+(\tfrac{1}{5}+\dots+\tfrac{1}{8})+\dots$$

may be shown to be divergent; and the series

$$1-2+3-4+5-6+\dots$$

to be $-\infty$ or $+\infty$, according to the method of bracketing the terms.

3. The ground is thus prepared for detailed consideration of the series

$$1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{1}{6}+\dots,$$

which is equal to

$$\frac{1}{2}+\frac{1}{3.4}+\frac{1}{5.6}+\frac{1}{7.8}+\dots$$

$$>\frac{1}{2}+\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\dots$$

* Abstract of a paper read at the General Meeting of the Mathematical Association, April 13, 1944.

$$= \tfrac{1}{2} + \tfrac{1}{2}(\tfrac{1}{3} - \tfrac{1}{5} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{7} - \tfrac{1}{9} + \ldots)$$
$$= \tfrac{1}{2} + \tfrac{1}{6} = \tfrac{2}{3}$$
$$> \cdot 65.$$

It is also equal to

$$1 - \frac{1}{2 \cdot 3} - \frac{1}{4 \cdot 5} - \frac{1}{6 \cdot 7} - \ldots$$
$$< 1 - \frac{1}{2 \cdot 4} - \frac{1}{4 \cdot 6} - \frac{1}{6 \cdot 8} - \ldots$$
$$= 1 - \tfrac{1}{4}(1 - \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{3} - \tfrac{1}{4} + \ldots)$$
$$= \tfrac{3}{4} = \cdot 75.$$

Hence the sum of the series is ·7, to one place. The diagram of the series is shown in the upper part of Fig. 2 (*a*).

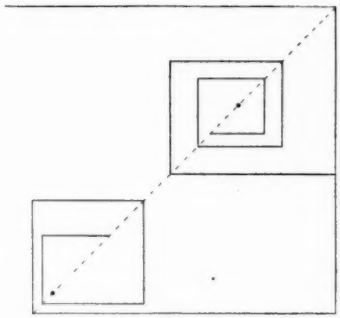


FIG. 2 (*a*).

4. The crucial peculiarity of this series, in which the change of minus to plus signs gives a divergent series, is that systematic alteration of the order of the terms involves an alteration of the sum.

Thus taking two negative terms to each positive term we obtain :

$$(1 - \tfrac{1}{2} - \tfrac{1}{4}) + (\tfrac{1}{3} - \tfrac{1}{6} - \tfrac{1}{8}) + (\tfrac{1}{5} - \tfrac{1}{10} - \tfrac{1}{12})$$
$$= (\tfrac{1}{2} - \tfrac{1}{4}) + (\tfrac{1}{6} - \tfrac{1}{8}) + (\tfrac{1}{10} - \tfrac{1}{12})$$
$$= \tfrac{1}{2}(1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \ldots),$$

or half the original sum. The lower part of Fig. 2 (*a*) shows the consequence of taking three negative terms to each positive term.

By contrast, if we take two negative terms to each positive term in the geometric progression

$$12 \cdot 8 - 6 \cdot 4 + 3 \cdot 2 - 1 \cdot 6 + \cdot 8 - \cdot 4 \ldots,$$

we obtain

$$12 \cdot 8 - 8 + 3 \cdot 2 - \cdot 5 + \cdot 8 \ldots$$

as in Fig. 2 (*b*). The diagram ascends by a series of decreasing loops to the original sum.

Furthermore, we can re-arrange the terms of

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \ldots$$

to arrive at any sum whatsoever, the process being that of alternately just over-shooting and just under-shooting the mark. For the sum 1·2, the re-arrangement begins :

$$(1+\tfrac{1}{3})-\tfrac{1}{2}+(\tfrac{1}{5}+\tfrac{1}{7}+\tfrac{1}{9})-\tfrac{1}{4}+(\tfrac{1}{11}+\tfrac{1}{13})-\tfrac{1}{6}+(\tfrac{1}{15}+\tfrac{1}{17}+\tfrac{1}{19})-\tfrac{1}{8}.$$

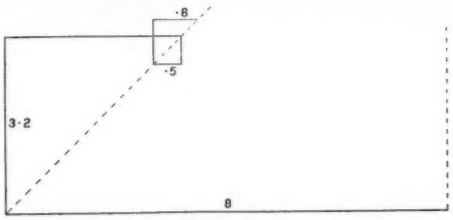


FIG. 2 (*b*).

5. The non-uniform convergence of the geometric progression as the common ratio approaches unity may be brought out by means of a succession of diagrams like Fig. 1, for common ratios $\pm\frac{2}{3}$, $\pm\frac{3}{4}$, etc. For positive common ratios the figures soon become unwieldy, so that numerical illustration is better. But for negative common ratios, all the figures are within bounds. The extreme cases can also be shown in figures : when $r=1$, the broken lines of the upper part of Fig. 1 become parallel, illustrating $1+1+1+\ldots$; while when $r=-1$, the diagram is a square, illustrating the oscillating series $1-1+1-1+\ldots$. N. M. G.

BUREAU FOR THE SOLUTION OF PROBLEMS.

THIS is under the direction of Mr. A. S. Gosset Tanner, M.A., 115, Radbourne Street, Derby, to whom all enquiries should be addressed, accompanied by a stamped and addressed envelope for the reply. Applicants, who must be members of the Mathematical Association, should whenever possible state the source of their problems and the names and authors of the textbooks on the subject which they possess. As a general rule the questions submitted should not be beyond the standard of University Scholarship Examinations. Whenever questions from the Cambridge Mathematical Scholarship volumes are sent, it will not be necessary to copy out the question in full, but only to send the reference, *i.e.* volume, page, and number. If, however, the questions are taken from the papers in Mathematics set to Science candidates, these should be given in full. The names of those sending the questions will not be published.

Applicants are requested to return all solutions to the Secretary.

GLEANINGS FAR AND NEAR.

1451. " Max . . . is a genius—a brilliant mathematician. . . . Write down on the black board any sum you like. Multiplication, if you like. Make it as difficult as you know how, and I guarantee that Max will write down the answer without a moment's delay." . . . Kirkwall . . . wrote a number containing twelve figures and, below it, another equally large. " Ask Max to multiply them," he said. " Make it more difficult," Ebben urged him. Kirkwall gasped, and then added the square-root symbol to one number.—Goodchild and Roberts, *The Prisoner's Friend*, p. 211. [Per Prof. E. H. Neville.]

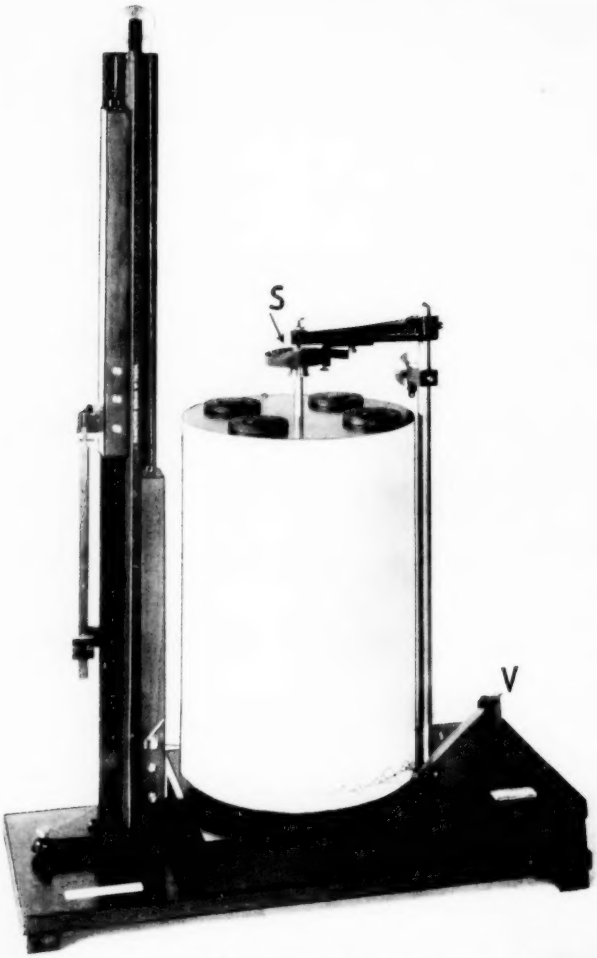


Photo by Messrs. G. Cussons, Ltd.

FIG. 1.

SIMPLE HARMONIC MOTION EXAMINED.

By E. J. Atkinson.

To the ordinary student simple harmonic motion is the motion of the bob of a swinging pendulum, of a mass at the end of an oscillating spring, or of a point on the rim of an oscillating fly-wheel; by definition, a body or particle is moving in simple harmonic motion if its acceleration is proportional to its distance from a fixed point, and directed towards that point.

The experiment* here described is designed to illustrate or to give reality to the definition. The case considered is that of the particle on the rim of a fly-wheel oscillating in a horizontal plane; the motion of the particle is plotted against a vertical velocity. The apparatus† illustrated in the photograph (Fig. 1), consists of a "Controlled-Fall" trolley which carries a marking brush to mark a sheet of paper fastened to a cylinder of which the disc is the base. The oscillation of the cylinder is controlled by the spring S, attached to the axis of the cylinder. With the controlled-trolley falling with uniform velocity the cylinder is released to oscillate, and the brush of the vibrator V, already in motion, will mark what can be called a "Time-Interval" trace upon the cylinder.

Examination of the traces will show that the amplitude Y_1Y_2 of the "combination trace" (Fig. 2) will be equal to the length y_1y_2 of the vibrator trace (Fig. 3). If the mean position line be drawn through the vibrator trace

Fig. 2.

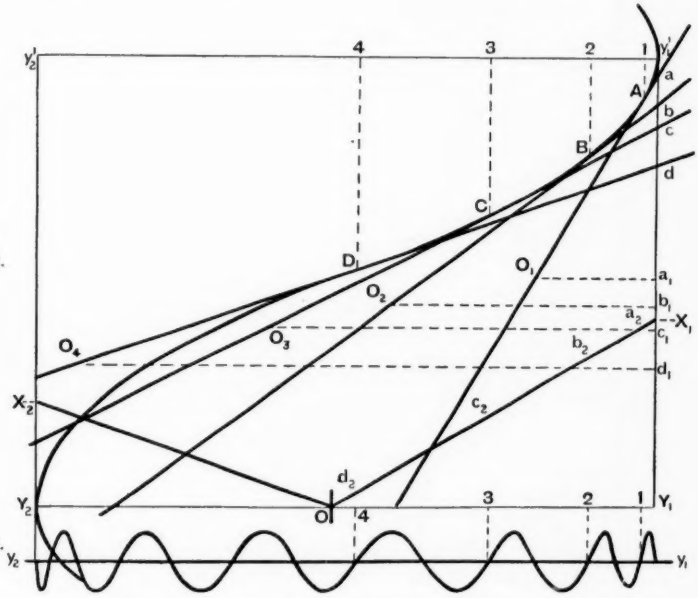


Fig. 3.

* Demonstrated to the London Branch, January 1936.

† Made by Messrs. G. Cussons, Ltd., Manchester, 7.

the displacement at each period (and possibly half-period) interval is obtained. Since the " combination trace " is the result of two velocities, one of which is constant, a measure of the variable velocity at any point can be obtained as the length of the horizontal side of the triangle formed with one side a fixed vertical length representing the known velocity, a side drawn parallel to the tangent and the third side horizontal (*i.e.* parallel to the direction of the variable velocity). The " vibrator " trace is applied to the " combination trace " and A, B, C, D, etc., the positions after successive equal intervals of time, are marked. The tangents aO_1, bO_2, cO_3 are drawn; the vertical velocity of the falling trolley is represented by the equal lengths aa_1, bb_1, cc_1, etc., and thus a_1O_1, b_1O_2, c_1O_3, etc., represent measures of the velocity of the particle at A, B, C, etc.

Using these lengths, a velocity time-graph (Fig. 4) is plotted. The tangents

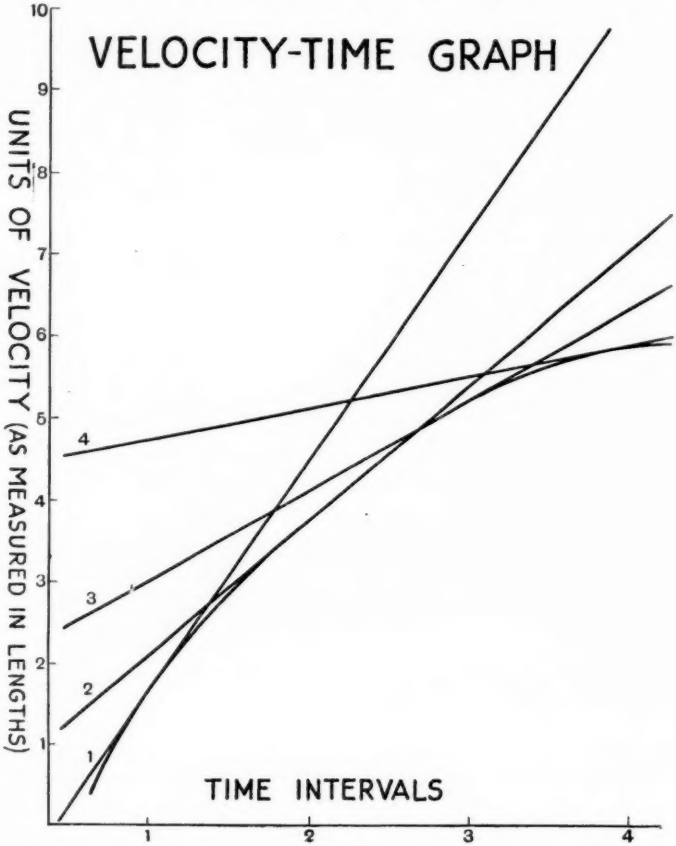


FIG. 4.

to this are drawn at the respective time interval 1, 2, 3, etc., and measures of the acceleration at each instant obtained. The acceleration measures are transferred back to a horizontal axis on to the original combination trace as $1a_2$, $2b_2$, $3c_2$, $4d_2$, and it is noted that $a_2b_2c_2d_2$ lie on a straight line X_1O cutting this horizontal axis (*i.e.* it is zero) at O, the centre of the oscillation. By applying this method on the other half of the curve the line OX_2 is obtained, and X_1OX_2 shows that the acceleration decreases uniformly to zero, then increases uniformly as the distance of the particle from the centre changes. Thus the fact so well known from theoretical considerations is given practical demonstration.

If by means of a vibrator a trace is made on the falling trolley, the true value of the constant velocity can be calculated and hence the true value of the variable velocity at any particular point may be deduced.

E. J. A.

PAPER FOR BOOKS.

RECENT statements concerning an increase in the allocation of paper to publishers appear to imply that the difficulty of obtaining school textbooks should soon be much diminished; it therefore seems desirable to draw the attention of members of the Association to the following letter from the President of the Publishers' Association (published in *The Times*, October 19, 1944):

"When the President of the Board of Trade announced the recent increase in the allocation of paper for books he allowed it to be understood that the increase brought the total approximately from 22,000 to 40,000 tons per annum. In a more recent statement on paper for school books the Minister of Education has stated that 'a substantial additional tonnage of paper has now been allocated which should go a considerable way towards meeting the most urgent cases of shortage'. These statements give the impression that there will now be a substantial increase—indeed an increase of about 80%—in supplies of books and I am concerned at the great expectations which must be aroused, in particular in the minds of schoolmasters, and at the effect which the inevitable failure to meet them will have upon publishers' good faith. Apart from a special allocation for the books required for the Services' post-war education scheme, none of which of course will go to the schools, the amount of paper licensed to publishers by quota has been increased by approximately 1,300 tons per annum, to be used mainly for export, and the amount of the reserve for meeting special needs has been increased by 600 tons per annum, of which one-fifth may be expected to find its way into school books—a total increase from approximately 22,000 to 24,000 tons per annum. While I do not wish to look even this small gift horse in the mouth, it is impossible not to recall that by a recent order of the Ministry of Supply 'overmakes', which by the custom of the paper trade are supplied to publishers to replace damage in transit and spoilage in manufacture, are no longer permitted, so that what has now been given by one Governmental hand does little more than replace what has recently been taken away by another. *It is not, therefore, to be expected that there can be any appreciable increase in supplies of school books.*

Yours faithfully,

R. J. L. KINGSFORD (sgd.),

President,
The Publishers' Association.

SOME PRINCIPLES AND PRACTICAL POINTS IN PREPARING TEACHING TIME-TABLES.

By F. Sandon.

§ 1. *Introduction.*

A school time-table is one of the means by which we give opportunities to a *pupil*, a member of a *class* in a *school*, to study during a definite *time* (a school *period*) and *place* (a *desk* in a *room*: see below) a particular *lesson*, one of a prescribed *syllabus* in a *subject*, with the help of a particular *teacher*, member of the school *staff*. It thus involves a process of grouping the five variables—period, teacher, pupil, room and subject—according to certain desirable objectives and rules.

§ 2. *The Five Variables: Practical Considerations.*

There are a number of practical difficulties to be met.

(1) The pupil is usually one of a *class*. The class should, most conveniently from points of view alike of economy of teaching strength and of stimulus to the pupil, be one of a more or less homogeneous group of suitable size. It is possible, however, to have a more heterogeneous assemblage as in:

(*a*) certain forms of the Dalton Plan, in which pupils of various abilities and standards, and even studying different subjects or syllabuses, are under the general supervision of the same particular teacher at any particular time;

(*b*) in other forms of private study;

(*c*) in cases where a teacher is instructing two, or even more, different sections or groups more or less concurrently.

The last of these is not desirable or very practicable: the two former are only applicable or appropriate in certain somewhat limited cases. For our purposes, we shall take as our unit the *class*, which will consist of one or more pupils originally at the same general level and capable of keeping pretty well in step.

The class, however, is not altogether a satisfactory unit, for its size and constitution are in some cases rather indefinite. For different purposes, it changes its membership: now it consists, say, of the boys studying Latin out of a particular form of boys and girls studying a second language, with the result that its size decreases; now again it may consist of all the boys from a number of forms, for the purpose of field games, or all the pupils from a group of forms who study a particular subject, so that the size of the class may increase; yet again, the size may remain much the same but a whole group of forms may be reshuffled into sets for particular purposes. Thus a school organised primarily on ability in classics may have all its middle school simultaneously reorganised for, say, mathematics.

A school consists for our present purpose of a number of classes working in a definite place with a teaching force of a number of instructors more or less permanently associated with it.

(2) The *time* is usually an interval that, for a particular school, is a definite *period*, one of a number of approximately equal periods, forming a cycle that is usually a week long, though the recurrence may be daily, terminal, or annual, as for certain cyclical events in the school life and routine that are not so closely associated with the teaching process as those with which we are now concerned. The period is of definite duration, but for particular subjects of study or pupil activity two or more periods may be arranged to follow without any gap to give double, triple, or *n*-ple periods. On the whole, for any one school, all periods begin, and finish, simultaneously for all the classes of the school.

We may note here that, as so often in education, the technical term has by no means the ordinary signification that it usually bears. A " period " in mathematical physics is, of course, the duration of the whole cycle, and not the duration of a specific part of it. Similar confusions may arise in respect of the use of such words as " session ", which means both a part of a school day, and a part, or the whole, of a school year. " Form " and " Report " have at least three distinct meanings to school teachers. " Left " has two contradictory meanings, sometimes indicating what the actuary calls " survivors " and sometimes what he calls " exitants ".

(3) The *place* consists primarily of a building of *rooms* usually referred to as the school. But annexes—other buildings, playing fields, etc.—are often found. They are usually close to the main building, as otherwise great waste of time is involved in the mere physical transport (in addition to the awkward question of the teachers' responsibilities for accidents in transit), and double-periods, or even longer, have to be devoted to obtain a single period of instruction.

For convenience, we shall refer to the place which a class, as above defined, occupies for a particular period as a *room*.

(4) The *subject* is a field of study consisting of a number of units—*lessons*—put together in a syllabus. The subject is usually, but not necessarily, developed logically according to the academic disciplines, as modified by pedagogic principles, but it may be a " project " or other classification that will provide a sequence that appears to have pedagogic value. A subject is, for all that, vague, and a lesson given by one teacher in, say, geometry may be so alike in, say, subject matter, method of presentation or development, or of emphasis, as to be classified alternatively as a lesson in mensuration, or logic, or drawing, or algebra, or geography, and so on.

(5) The *teacher* is an individual who possesses qualifications that appear valuable for the lesson for the pupil in question. Usually time-table difficulties arise because of the limitations of the qualifications, by way of academic knowledge, physical skill, pedagogic ability, and so on, that make it desirable to restrict the studies, etc., of the pupil at the hands of that teacher. A teacher may, however, be versatile and capable of giving the pupil all that is necessary, when time-table difficulties are less likely to arise.

For convenience later, we shall adopt the following notations :

Classes :	$A, B, C, D, \ldots$.
Periods :	1, 2, 3, 4, ...
Rooms :	I, II, III, IV, ...
Subjects :	$\alpha, \beta, \gamma, \delta, \ldots$.
Teachers :	$a, b, c, d, \ldots$.

It is to be noted that mathematically any one of these five variables may be replaced by any other.

§ 3. *Mathematical Arrangements of the Five Variables.*

From the above brief summary of the elements with which the time-table constructor has to deal, we see that the practical difficulties arise through the indefiniteness of any of the first four variables and the limitations of the fifth, the teacher. In the simplest case, where the class is sharply delimited throughout the range of the cycle of " periods ", and it is limited physically to the confines of a single room and receives all its instruction at the hands of one particular teacher, the time-table construction consists simply in rotating as required (according to statute law—as for the first half-hour each day for scripture in public elementary schools—or to the recommendations of school hygiene—arithmetic early in the day, physical activity at the end—or what-

ever other requirements may be prescribed) the subjects and the periods. This is a simple problem of permutations: there are, of course, 3! or 6 different arrangements, if no restrictions are to be applied, in the case of three subjects and three periods.

The case next in order of complexity is where we have three variables to deal with. We may, *e.g.*, suppose that the class and the teacher are fixed, and the subject, the period, and the room have to be brought into some particular arrangement. Or it may be that the class and the room are fixed, and we have a simple form of specialist instruction, and rotate period, subject and teacher. This presents more difficulty. If we have n of each of the three variables, none to be repeated in the cycle, then the problem is that solved by Euler in his Latin Square, and now used by the Fisher school of analysis of variance for appropriate experimental design. Thus for three of each variable we might have the following solution, Table I (quoting from the first of such tables given in Fisher and Yates, *Statistical Tables* (Oliver & Boyd, 1938,) p. 46, Table XVI, i). There are 12 three by three Latin Squares. We observe that the following kinds of requirements are satisfied:

(*a*) every class has each specialist teacher once in a cycle;
(*b*) no teacher is with two classes at once;
(*c*) every class has a teacher every period.

Similar rules can be noted by rearranging periods, teachers and classes.

TABLE I.

Class.	Period.				
	1	2	3	4	...
A	*a*	*b*	*c*	*a*	...
B	*b*	*c*	*a*	*b*	...
C	*c*	*a*	*b*	*c*	...
...	...	...	...	...	...

TABLE II.

Class.	Period.			
	1	2	3	4
A	*a*I	*b*II	*c*III	...
B	*b*III	*c*I	*a*II	...
C	*c*II	*a*III	*b*I	...

Yet a further stage of complexity arises when we have four variables to be considered. Let us suppose that each value of each of the four has to be associated with each value of the other three once and once only in a cycle. Thus, quoting the example given by R. A. Fisher, *The Design of Experiments*, (Oliver & Boyd, 1935, p. 90), we have the case given in Table II. This is known as a Graeco-Latin Square. There are, for our case of three values of each variable, 72 three by three Graeco-Latin Squares.

In the tables just given, we have what are rather artificial cases for practical purposes, but they illustrate the principles of solution that have to be considered, and the nature of the solutions that arise.

§ 4. *Practical Modifications of these Arrangements.*

In actual practice, however, we do not work with equal numbers of teachers, rooms, classes, subjects and periods, though the first three are of the same general numerical value, and the fourth is of much the same order of magnitude. The necessity of n-ple periods, and similar considerations, make it impossible to subdivide the cycle into series each containing a number of periods approximately equal to the number of the four variables already considered. The problem gets rather a long way removed from the standard mathematical one that we have just considered. If the administration has been successful, there will not be more classes than teachers, or classes than rooms. Actually a further administrative point arises on an aspect that we will not otherwise consider: a room contains a certain number of desks, a class a certain number of pupils, and the number of desks must be not less

than the number of pupils, both in each class and in the school as a whole: if the classes are not equal in size, and classes use various rooms in turn, more desks will be needed in the school than there are pupils in the school. By "desk" we mean not merely a "box with a seat", but also such "places" as "stools" in a lab., "vices" in the workshop, or even "places" in a games team. But even if these considerations are satisfied, there will remain many other modifying features, and these must be allowed for in a definite order if a satisfactory time-table is to be obtained. The time-table can be regarded as a means of translating the available frequency two-way table, such as Table III (where the Arabic numbers are the number of periods per cycle) into the desired frequency two-way table such as Table IV through the working time-table such as in Table V (Arabic numbers now ordinals).

TABLE III.

Class.	Teacher.			
	a	*b*	*c*	...
A	α, 5	β, 3	γ, 3	...
B	α, 4	β, 3	γ, 3	...
C	δ, 6	β, 7	ϵ, 2	...
...	...	...	...	...

TABLE IV.

Class.	Subject.			
	α	β	γ	...
A	5	3	3	...
B	4	3	3	...
C	0	7	6	...

TABLE V.

Class.	Period.			
	1	2	3	...
A	$a\alpha$I	$b\beta$II	$c\gamma$III	...
B	$b\beta$II	$a\alpha$I	$a\alpha$I	...
C	$c\gamma$III	$c\gamma$III	$b\beta$II.	...
...	...	...	...	...

The translation cannot be done directly because of the necessity, as above indicated, for double periods, the desirability of, say, one period per day for a particular subject, the shortage of laboratory accommodation, the convenience of visiting teachers, the regrouping of the school into sets, and so on. It may be noted in connection with this last reorganisation, and with similar variations in the composition of the "class", that the analysis tables such as III and IV become themselves complicated. It is probably most convenient to have for each class, say, a number of columns, one for the form subjects and two for the two alternative subjects (more columns if more alternatives at any one time).

§ 5. *Steps in Building a Time-Table.*

The practical steps in obtaining the time-table (period and class as marginal captions as in Table V) are conveniently as follows.

(1) Make certain that the administrative limitations are cleared—that there are enough teachers and rooms for each of the classes (and similarly enough desks for the pupils), and fix the length and numbers of periods in the cycle.

(2) Prepare the analysis table such as III above.

(3) Work with the teacher as the unit. As we have pointed out, he is the only discrete individual of our variables. We may conveniently represent him by a "map-marker", coloured drawing pin, shaped push-pin or similar device, the particular colour or shape representing the particular teacher. He alone (except perhaps the room, which is not a practicable basis for time-table making) cannot be subdivided or combined, and the fundamental prin-

ciple is that the teacher in any one period cannot be in two rooms, or teach two classes, or two subjects, though, of course, if the formal Form is reclassified into sets it may not immediately look like this on the time-table.

(4) First assign teachers to classes, groups of classes, etc., for the following special cases.

(*a*) Fill up all the blanks determined by external circumstances (*e.g.* broadcast lessons at stated times).

(*b*) Assign, as convenient, the provision for triple and for double periods. If this is not done at an early stage, it will probably be found impossible later on.

(*c*) Where there are visiting teachers, so that the number of teachers available differs throughout the cycle, or where there is a system of regrouping classes, estimate for each period of the cycle the number of teachers available, and see that the regrouping, etc., is at such times that the numbers of classes does not exceed the number of teachers available for each period.

(*d*) Similarly, where the number of " rooms " available fluctuates, whether class rooms or games fields, gym or bath accommodation, estimate per period the number of rooms available, and see that the number of classes in any regrouping is not more than the number of rooms at any one time.

(*e*) For periods when there is any regrouping of forms into sets, fill up these periods with the desired allocation to the reshuffled groups.

(5) If attention has been paid to all cases where the number of classes might have otherwise worked out to give more than the number of teachers or rooms available, then the time-table can be plotted with the three variables, period, class, and teacher, with the teacher as the entry in the table.

(6) After the double-periods and other bracket cases (whether horizontal over time, or vertical over classes) have been dealt with, then the periods per class for any particular teacher can be allotted as uniformly as possible through the cycle, due regard being paid to any special legal, hygienic, or pedagogic needs. It will probably be found helpful to have written notes available, consolidating all the various special limitations, restrictions and suggestions. It is usually convenient first to finish off the time-tables of those teachers whose time-table is most nearly completed as a result of the earlier steps suggested above.

(7) " Jamming " may occur, just as it does in completing a Latin Square using random numbers. With, say, twelve classes and, say, forty periods, *i.e.* with approximately 500 entries to be inserted, it might be thought that the numbers of possible time-tables is so immense that there would always be many alternatives at any stage. In actual practice, considerations such as those we have indicated at various places above so limit the possibilities that in the final stages jamming is almost certain to arise. The use of pins permits without excessive confusion of the experimental modification of the table over as many of the variables as may be necessary.

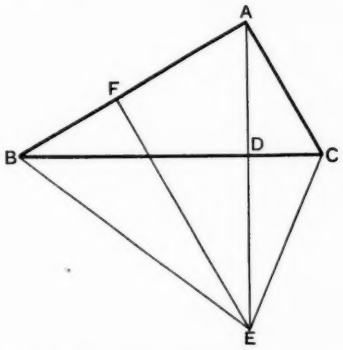
(8) When the time-table is completed for the three variables mentioned, the other two variables—subject and room—can be fairly easily fitted in. The Graeco-Latin rotation of rooms is seldom necessary. In practice, rooms are determined by the class (which uses its form room as much as possible) or by the subject, and thus largely the teacher, since in some cases of specialist instruction (as, *e.g.*, in science) the necessary equipment and apparatus is available only in a specific room. In addition to the summary table, as Table IV, a teacher table (teacher and period in the margins, classes, rooms, and subjects in the entries) and a room table (room and period in the margins, teacher, class and subject in the entries) will be found a useful compilation, readily extracted from the main time-table.

FRANK SANDON.

MATHEMATICAL NOTES.

1746. *A proof of Pythagoras' theorem.*

The triangle BAC is right-angled at A. Draw AD perpendicular to BC and produce it to E, where $AE = BC$. From E draw EF perpendicular to AB and join EB, EC.



Then the triangles AFE, BAC are congruent, and hence $AF = AC$, $EF = AB$. Now the sum of the triangles ABE, ACE is the quadrilateral $ABEC$.

Hence $\quad \frac{1}{2}AB \,.\, FE + \frac{1}{2}AC \,.\, AF = \frac{1}{2}BC \,.\, AE,$

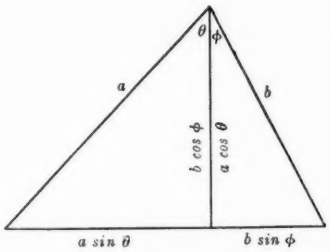
or $\quad \frac{1}{2}AB^2 + \frac{1}{2}AC^2 = \frac{1}{2}BC^2.$

N. KAILASAMAIYER.

1747. *On Note* 1680.

Instructor Captain G. A. Clarkson, R.N., and Mr. C. E. Kemp point out that this proof of the addition theorem is fairly well known and can be found in H. E. Piggott, *Plane Elementary Trigonometry*, 3rd edition, 1925 (Constable), p. 165.

Captain Clarkson adds: I sometimes, with a good class, put it in the one-line form:

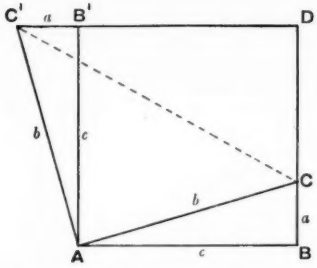


$$\tfrac{1}{2}ab \sin(\theta + \phi) = \Delta = \tfrac{1}{2}a \sin\theta \,.\, b \cos\phi + \tfrac{1}{2}a \cos\theta \,.\, b \sin\phi.$$

1748. *Pythagoras' theorem.*

The following American proof is very far from new, but does not seem to

be well known.



Swing the triangle ABC through $90°$ to $AB'C'$. Then

$$\text{area } ACDC' = \text{area } ABDB';$$

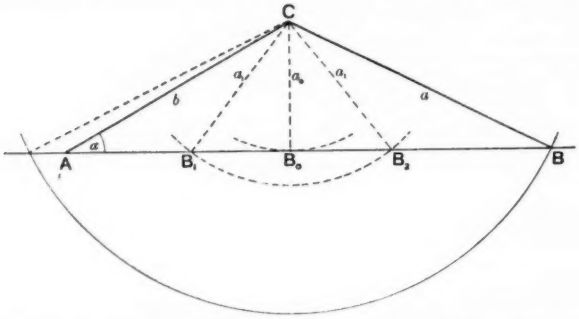
that is, $$\tfrac{1}{2}b^2 + \tfrac{1}{2}(c+a)(c-a) = c^2,$$

or $$a^2 + c^2 = b^2.$$ G. A. C.

1749. *The fourth case of congruence.*

In the Geometry textbooks generally used at secondary schools in this country the case—hypotenuse—side—right angle (*hsR*)—is mostly called the fourth theorem of congruence. This is amazing (and different from the custom in other countries) because actually the fourth theorem would be *SsA*, *i.e.* Side—side—the Angle opposite the greater of the two sides, so that *hsR* is only a special case of the real fourth case of congruence.

I think it would be instructive and useful to deal with some problems of construction of triangles, first of all with the so-called ambiguous case where the given angle is opposite the smaller of the two sides. Construct a triangle from $a_1 = 5$ cm., $b = 8$ cm., $\alpha = 30°$; then vary the length of the side a, *e.g.* $a_0 = 4$ cm., finally $a = 9$ cm.; the triangle ABC is uniquely determined.



Of course this general case *SsA* does not occur as often in applications as the special case *hsR*. But it is not compatible with the high standard of English Geometry textbooks, with their thoroughness, exactitude, and comprehensiveness, compared with the corresponding books used on the continent as far as I have known them in my practice extending over twenty-five years.

S. WEIKERSHEIMER.

1750. *The angular momentum of a lamina.*

Let a rigid lamina, of mass M and centre of mass G, move in its own plane. At a given instant let ω be the angular velocity and I the instantaneous centre; then the velocity of G is $GI \cdot \omega$ in the direction GX, say. With GX, GI as axes of coordinates, let P be the point (x, y) in the lamina. The moment of inertia about G being Mk^2, that about P is $M(x^2+y^2+k^2)$, and the angular momentum about P is $M(GI \cdot y + k^2)\omega$, the sense of measurement being the same as that of ω.

Hence, if the angular momentum about P is equal to the product of ω and the moment of inertia about P,

$$x^2 + y^2 - GI \cdot y = 0.$$

Thus the point P lies on the circle which has GI as diameter.

This result, even if not new, does not seem to be well known. It may be useful to teachers in overcoming the tendency of beginners (and others!) to take the angular momentum about any point to be the product of the angular velocity and the moment of inertia about that point. S. L. G.

1751. *Pythagoras and an extension.*

The euclidean proof consists essentially in the demonstration of the equality of area of a square and a rectangle, and this is effected by the use of congruent triangles. In such a well-worn field there can hardly be any real novelty of proof, but alternatives to the use of congruent triangles have been found and it is possible to modify and extend the methods.

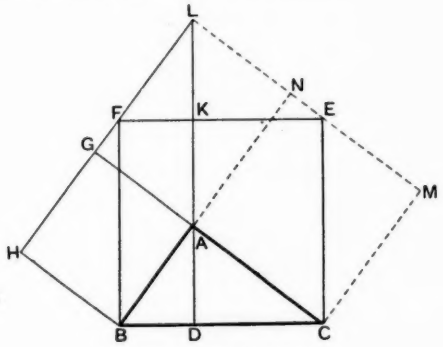


FIG. 1.

In Fig. 1 the squares $ABHG$ and $BCEF$ are drawn and the altitude KAD. We want to prove that the square BG and the rectangle BK are equal in area. First we deduce from the congruence of the triangles BHF and BAC that HF is perpendicular to HB and therefore along the same line as HG. Let it be produced to meet DAK produced in L. Then by the usual theorems on areas between parallels

$$\text{square } BG = \text{parallelogram } BL = \text{rectangle } BK.$$

Similarly the square $ACMN$ is equal to the rectangle CK and the proof can be completed.

The points to be noticed for the extension are that L is the intersection of HG and MN and that AL is equal to BF.

In Fig. 2 is shown a means of drawing a parallelogram on a given base equal to the sum of two given parallelograms. The given base is BC, $ABHG$ and $ACMN$ the given parallelograms; their sides HG and MN are produced to meet in L. The line LA is produced to cut BC in D, and DK marked off equal to LA. (This can be done in either direction, but to save crossing of lines the opposite direction to that of Fig. 1 is taken in Fig. 2.) Finally the parallelogram $BCEF$ is drawn with BF equal and parallel to DK. We now have by equal parallelograms :

$$BG = BL = BK,$$
$$CN = CL = CK.$$

By addition the sum of the original parallelograms is $BCEF$ on the given base BC.

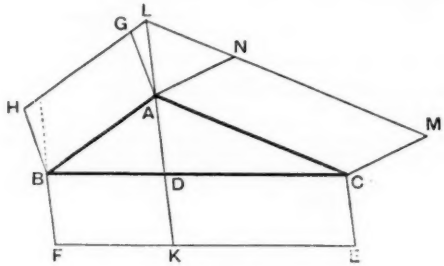


FIG. 2.

It is clear that there are conditions restricting the length of BC, but in any case in which they are not satisfied an easy modification of the construction is practicable. B. E. LAWRENCE.

1752. *The solution of linear differential equations of the second order.*

Since the method of solving a linear differential equation of the second order by means of a Riccati equation (Note 1578) appears to have aroused some interest, I describe below another method which is likewise little known and not to be found in the textbooks.

Given the equation $y'' + Py' + Qy = R$, we require to determine functions u, v such that $y'' + Py' + Qy = (D - u)(D - v)y$, where $D = d/dx$.

Now $(D - u)(D - v)y = y'' - (u + v)y' + (uv - v')y$, so that we have $u + v = -P$, and $uv - v' = Q$.

Eliminating u, we obtain for v the equation $v' + (v + P)v + Q = 0$.

This is precisely the Riccati equation on which the previous method of solution depended. If any particular function v can be found to satisfy it, a corresponding value of u is then determined, and we may proceed to solve the given linear equation by the standard method of "factorising the operator".

Consider, for example, the equation $y'' - (1 + 2/x)y = 1$. Here $u = -v$, and the corresponding Riccati equation is $v' + v^2 = 1 + 2/x$. It will be observed that a particular solution of this equation is $v = 1 + 1/x$. Thus, writing the second-order equation in the form $(D + v)(D - v)y = 1$, we see that it may be replaced by the pair of first-order equations $dz/dx + vz = 1$, $dy/dx - vy = z$, where $v = 1 + 1/x$. From the first of these we obtain the complete primitive z, whence a second quadrature gives us the general solution for y. L. ROTH.

1753. *The asymptotes of the hyperbola* $x^2/a^2 - y^2/b^2 = 1$.

The Rev. T. H. Ward Hill's article, "Asymptotes in Parametric Co-ordinates" in the *Gazette*, May, 1943, reminded me of a method of determining the asymptotes of the hyperbola $x^2/a^2 - y^2/b^2 = 1$, which I have not seen in elementary textbooks on conic sections, and which may be of some interest to readers.

The equation of the tangent at the point $(a \sec \theta, b \tan \theta)$ on the hyperbola

$$\frac{x \sec \theta}{a} - \frac{y \tan \theta}{b} = 1,$$

that is,

$$\frac{x}{a} - \frac{y \sin \theta}{b} = \cos \theta.$$

Now let $\theta \to \frac{1}{2}\pi$. The point $(a \sec \theta, b \tan \theta)$ passes to infinity on the curve and the tangent becomes an asymptote with the equation $x/a - y/b = 0$.

If $\theta \to \frac{3}{2}\pi$, the asymptote $x/a + y/b = 0$ is obtained.

The position of the curve relative to the asymptotes is then easily determined by comparison of the ordinates

$$\pm b\sqrt{\left(\frac{x^2}{a^2} - 1\right)} \text{ and } \pm \frac{bx}{a}$$

for points with the same abscissa on the curve and on the asymptote.

J. DEANS.

1754. *The sign of* ρ *and related topics.* (*Math. Gazette*, XVII, p. 77.)

Professor Piaggio's article. But what about the signs of ds and of $d\psi$ taken separately? There are some queer snags. Following de la Vallée Poussin (I, p. 240), there are two conventions for the sign of ds, for cartesians in which ds follows dx, and for polars in which ds follows $d\theta$. Also the tangent is to be given a forward direction (p. 241), namely that of ds. Take a parabola referred to its axis and directrix; by the first convention the direction of s is reversed at the vertex, and in the upper part of the curve is opposite to what it is under the second convention. ψ is discontinuous either way; at the vertex under convention (1), at the points of contact of tangents from the origin under convention (2). Does the very existence of $ds/d\psi$ depend on a convention?

Following the second convention, and a counterclockwise rotation rule for angles, $rd\theta/ds$ is always positive and hence $0 < \phi < \pi$; therefore p is always positive, that is, the forward direction of p is from the origin to the tangent and its slope is $\psi - \frac{1}{2}\pi$. Hence the equation of the tangent is always

$$x \sin \psi - y \cos \psi = p,$$

by convention. Differentiating, the normal is always

$$x \cos \psi + y \sin \psi = dp/d\psi,$$

and the normal has a forward direction $+\frac{1}{2}\pi$ ahead of the tangent, even if the student does not happen to draw the well-known figure referred to by Professor Piaggio. If ρ being positive means that the centre of curvature lies in the forward direction of the normal from the point of contact, its polar coordinates referred to the latter are $(\rho, \psi + \frac{1}{2}\pi)$, so that

$$\xi = x - \rho \sin \psi, \quad \eta = y + \rho \cos \psi \text{ always.}$$

Again, the equations $\cos \psi = dx/ds$, $\sin \psi = dy/ds$ are fortunately unaffected

by the convention, except in exposition, so that the equation of the normal satisfied at the point of contact, quickly leads to $ds/d\psi = r dr/dp$, and both are positive in the above direction. Differentiating again, the normal to the evolute is

$$-x \sin \psi + y \cos \psi = d^2p/d\psi^2.$$

This makes an angle π with the tangent, and so $d^2p/d\psi^2$ is directed away from the tangent. Hence $\rho = p + d^2p/d\psi^2$, directed as before.

On the whole, this one process brings out all the usual results, and the inter-connection of signs. Draw the forward direction of the normal at an angle $+\frac{1}{2}\pi$ with that of the tangent. If the curve is convex to this direction, then either the curvature is negative, and the formulae are left to look after themselves; or the formulae are forcibly reversed in sign and the curvature is signless. Compare de la Vallée Poussin (p. 260), whose method of discrimination is not too clear near points of contact of tangents from the origin.

C. L. WISEMAN

1755. *On phase angle.*

The formula for b^2/a^2 in the article "Determination of Phase Angle ..." in the October 1943 *Gazette* (p. 157, l. 5) admits of some simplification.

Dividing numerator and denominator by $\sqrt{\{(1-R^2)^2 + 4R^2 \cos^2 \phi\}}$ we obtain

$$\frac{b^2}{a^2} = \frac{1 + R^2 - \sqrt{\{(1-R^2)^2 + 4R^2 \cos^2 \phi\}}}{1 + R^2 + \sqrt{\{(1-R^2)^2 + 4R^2 \cos^2 \phi\}}}.$$

Now $(1-R^2)^2 + 4R^2 \cos^2 \phi = (1+R^2)^2 - 4R^2 \sin^2 \phi$

$$\leqslant (1+R^2)^2,$$

and thus this expression can be written as $(1+R^2)^2 \cos^2 \psi$.

The relation between ψ and ϕ reduces to

$$2R \sin \phi = (1+R^2) \sin \psi, \quad \text{.....(i)}$$

and the expression for b^2/a^2 now simplifies to $\tan^2 \frac{1}{2}\psi$; hence

$$\tan \tfrac{1}{2}\psi = b/a.$$

From (i), $$\sin \phi = \frac{1+R^2}{2R} \cdot \frac{2ab}{a^2+b^2},$$

or denoting the ratio of the major axis to the minor axis by r, that is, $r = a/b$,

$$\sin \phi = \left(R + \frac{1}{R}\right) \Big/ \left(r + \frac{1}{r}\right).$$

We can now calculate ϕ from known values of R $(= B/A)$ and r $(= a/b)$, or ϕ can be very simply determined by the use of a nomogram.

C. E. A. BURNHAM

1756. *Howlers.*

The common chord of two circles is the line of centres.

The line joining the vertex of a triangle to its centroid is the common chord of two excircles.

If two orthogonal circles are drawn their common chord forms a harmonic range.

If two inverse points with respect to a circle can be drawn the circle is orthogonal.

The radical axis of two circles is a line drawn from a point outside the circles perpendicular to the line of centres.

The roots of the equation $ax+by=1$ are $[-b\pm\sqrt{(b^2+4a)}]\div(-a)$.
The sum of the roots of the equation $ax^2+by^2=1$ is $-by/ax$.

$$5x^2(1-2x^3)^{-1}=5x-10x^4.$$

If $v=90^\circ-u$, then $\tan v=\tan(90^\circ-u)=-\cot u$. Hence $v=-1/u$.

To solve the equation $2\cos(5x+17^\circ)=\cos 143^\circ\ 15'$. We have

$$\begin{aligned}2\cos 5x&=\cos 143^\circ\ 15'-2\cos 17^\circ\\&=0{\cdot}8023-1{\cdot}9126\\&=\bar{1}{\cdot}1103.\end{aligned}$$

Hence $\cos 5x=\bar{2}{\cdot}5551$, and so $5x=56^\circ\ 19'$.
If $\cos 5x-\cos x=\sin 2x$, $\cos 5x=\sin 2x\cos x$.
These are not schoolboys' howlers ; they are all taken from a batch of less than twenty scripts of university students sitting at the end of their first year.
In another examination at a higher standard and taken by a much larger number of candidates, exactly thirty per cent. of the candidates stated that

$$\sqrt{(3x-x^2-2)}=-\sqrt{(x^2-3x+2)}.$$

AN EXAMINER.

1757. *Reduction of the general conic when $h^2\neq ab$.*

The following method of relating a central conic to its centre and axes has the merit of being easily followed by students.
If the conic is

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

its axes are parallel to $h(x^2-y^2)=(a-b)xy$, which we solve, obtaining

$$(x-\alpha y)(\alpha x+y)=0.$$

This being so, the conic must be expressible in the form

$$\lambda(x-\alpha y+l)^2+\mu(\alpha x+y+m)^2+\nu=0.$$

First we find λ and μ from

$$\lambda+\mu\alpha^2=a, \quad\ldots\ldots\text{(i)}$$
$$\lambda\alpha^2+\mu=b, \quad\ldots\ldots\text{(ii)}$$
$$2\alpha(\mu-\lambda)=h. \quad\ldots\ldots\text{(iii)}$$

These will of course be consistent, and (i) and (ii) give $\lambda+\mu=(a+b)/(1+\alpha^2)$, so the solution is immediate. Next determine l and m from

$$l\lambda+m\mu\alpha=g,$$
$$-l\lambda\alpha+m\mu=f,$$

and finally,

$$-\nu=\lambda l^2+\mu m^2-c.$$

We thus quickly get our central conic in the form :

$$\frac{(x-\alpha y+l)^2}{-\nu/\lambda}+\frac{(\alpha x+y+m)^2}{-\nu/\mu}=1,$$

and our axes are automatically associated with their proper directions.

H. TODD.

1758. *The director circle of the general conic.*

Let $lx+my+n=0$ and $mx-ly+n'=0$ be two perpendicular tangents to the general conic. We require to find the locus of their point of intersection.
The condition that the first line should be a tangent is

$$Al^2+2Hlm+Bm^2+2Gln+2Fmn+Cn^2=0,$$

or, substituting for n,

$$Al^2 + 2Hlm + Bm^2 - 2(Gl + Fm)(lx + my) + C(lx + my)^2 = 0.$$

Similarly for the other tangent,

$$Am^2 - 2Hlm + Bl^2 - 2(Gm - Fl)(mx - ly) + C(mx - ly)^2 = 0.$$

Adding these two equations and dividing by $l^2 + m^2$, we get the equation of the director circle :

$$A + B - 2Gx - 2Fy + C(x^2 + y^2) = 0.$$

C. H. FISHER.

1759. *The circle of curvature in Cartesian coordinates.*

The usual definitions of the circle and radius of curvature at a point on a curve in terms of differential coefficients, implicitly or explicitly, often leave considerable doubt as to the interpretation at a multiple point and this article is an attempt to free the question from the difficulties involved.

By the standard theorem of the reversion of series the branches of a curve through any point P on it can be represented, after transference of the origin to P and for sufficiently near points, by groups of p equations (p being a positive integer) of the type

$$y = x(a + a_1 x^{\frac{1}{p}} + a_2 x^{\frac{2}{p}} + \ldots + a_{p-1} x^{\frac{p-1}{p}} + bx + b_1 x^{1+\frac{1}{p}} + \ldots),$$

where, if $p = 1$, the equation represents a single linear branch, and if $p > 1$, the group represents a group of p superlinear branches which cannot, strictly, be regarded independently at P. We shall refer to the branch or branches as a group of branches of order p : clearly more than one such group may pass through a point.

We propose as the definition of the circle of curvature " If through a point P on a curve there passes a group of branches of order p, and if through P a family of circles can be drawn for which any one has in effect r intersections ($p + 1 \leqslant r \leqslant 2p$) with the group, and which is such that one of the family has $2p + 1$ intersections or more, then that circle is defined as the circle of curvature at P for the group ". It will be found later that, when a circle of curvature exists, $r = 2p$ except for one particular case and in the particular case $r = p + 1$.

Any circle through P, $x^2 + y^2 - 2gx - 2fy = 0$, may be written as

$$y = f - (f^2 + 2gx - x^2)^{\frac{1}{2}}$$

for points near P, or, if $f \neq 0$,

$$y = -\frac{g}{f}x + \frac{1}{2f}\left(1 + \frac{g^2}{f^2}\right)x^2 + \ldots,$$

and if $-g/f = a$ the family of circles is constructed. In order that the special circle of the family may exist (the final condition of our definition), it is clear that the group of branches must be such that the coefficients $a_1, a_2, \ldots a_{p-1}$ are all zero : each circle of the family then has in effect $2p$ intersections at P with the group. For the circle of curvature, with $2p + 1$ intersections, we must choose f and g so that $\frac{1}{2f}\left(1 + \frac{g^2}{f^2}\right) = b$, or, since $-\frac{g}{f} = a$,

$$f = \frac{1 + a^2}{2b}, \quad g = -\frac{(1 + a^2)a}{2b}.$$

If x_1, y_1 are the coordinates of P with respect to the original axes, we have

therefore (provided the coefficients $a_1, a_2 \ldots a_{p-1}$ in the expansion vanish) a circle of curvature at P for the group, and the circle has centre

$$x = x_1 - \frac{(1+a^2)a}{2b}, \quad y = y_1 + \frac{(1+a^2)}{2b},$$

and radius ρ where $|\rho| = \left|\frac{(1+a^2)^{\frac{3}{2}}}{2b}\right|$. If we admit the idea of a circle of infinite radius, these results still hold for zero b as, for example, at a point of inflexion or of undulation. We have tacitly assumed that the expansion for y in terms of x is valid. If $\frac{1}{a} = 0$, we must interchange the variables and the expansion, for x in terms of y becomes $x = y(\alpha_1 y^{\frac{1}{p}} + \alpha_2 y^{\frac{2}{p}} + \ldots)$, corresponding to $f = 0$ from which we have been restricted.

If it happen that $a = +\iota$ or $-\iota$ so that $1 + a^2 = 0$, indicating that f and g are both zero, the previous analysis fails as we cannot then obtain an expansion for the circle. The point P is such that the tangent at P to a group of branches through P is a circular line. In the case of a curve whose equation is real, then the points P of this type occur in conjugate imaginary pairs with tangents giving the real foci, or coalesce in pairs in circular conjugate points which are again foci. Consider the intersections of the family of circles

$$x^2 + y^2 - \lambda(y - \iota x) = 0,$$

with the group of branches

$$y = x(\iota + a_1 x^{\frac{1}{p}} + a_2 x^{\frac{2}{p}} + \ldots).$$

A circle of the family consists of the lines $y - \iota x = 0$, $y + \iota x - \lambda = 0$. The first line has $p+1$ intersections with the group at P, and when $\lambda = 0$ the second has p. Hence the family of circles gives rise to a true circle of curvature, the point circle $x^2 + y^2 = 0$. For this family r in the definition equals $p+1$.

A group of branches of type $y = x(a + a_s x^{\frac{s}{p}} + \ldots)$, where a_s is not zero and $s < p$, has no true circle of curvature at (0, 0). If we take, however, a sequence of points $Q_1, Q_2, \ldots$ on a branch of the group (appointing a specified pth root) approaching (0, 0), the equation of the branch referred to axes through any point Q (h, k) of them becomes for points near Q

$$y + k = (x + h)[a + a_s(x + h)^{\frac{s}{p}} + \ldots],$$

i.e.
$$y = x[\{a + a_s h^{\frac{s}{p}}\left(1 + \frac{s}{p}\right) + \ldots\} + \{a_s\left(\frac{s}{ph^{1-\frac{s}{p}}} + \ldots\right) + \ldots\}x + \ldots],$$

or
$$y = x(a' + b'x + \ldots),$$

where a' remains finite and b' increases indefinitely as Q approaches (0, 0). We have then a sequence of true circles of curvature tending towards the limit $x^2 + y^2 = 0$, but this limiting point circle is not a circle of curvature of the sequence.

Taking this definition of the circle of curvature as a particular one of a certain family of circles (which we have shown are touching at a common point) and the definition of linear asymptotes as "If a family of parallel straight lines are such that any one of them has only $n - r$ $(r \geqslant 1)$ finite intersections with a curve of degree n, and if one (or more) of them has fewer than $n - r$, then such a line (or lines) is defined as a linear asymptote" (cf. Green,

Math. Gazette, Dec. 1926, p. 234), the linkage between the theories of curvature and of linear asymptotes by means of inversion is complete. From the known properties of inversion the inverse of a linear asymptote of a curve with respect to any point is a circle of curvature to the inverse curve at the point (parallel repeated asymptotes corresponding to circles of curvature of branch groups which touch) with the special case of an asymptote which is a circular line which becomes a circular line and a circular line tangential to the inverse curve at the centre of inversion, with the converse property throughout.

H. GWYNEDD GREEN.

1760. *On Note* 1669.

L. Hogben, in *Science for the Citizen*, provides a method for obtaining the formula $\tan\theta = \tan\phi \,.\, \sin\lambda$ by use of a solid model, simply constructed.

F. W. K.

1761. *The tangent to a conic.*

An Oxford and Cambridge Higher Certificate paper in July, 1942, contained as part of a question : " Prove that the equation of the tangent at the point (x_1, y_1) on the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

is $$x(g + x_1) + y(f + y_1) + gx_1 + fy_1 + c = 0.$$"

This less usual form seems an appropriate peg upon which to hang a plea for a more orderly presentation of equations generally. I have for several years suggested that my classes memorise the equations of tangents as

$$x_1 \,.\, x + y_1 \,.\, y = r^2,$$

or $$y_1 \,.\, y = 2ax + 2ax_1, \text{ etc.}$$

Similarly the question quoted would read :

$$\text{" . . . is } (g + x_1)x + (f + y_1)y + gx_1 + fy_1 + c = 0."$$

These (although perhaps not so neat typographically) do emphasise the properties of the lines in relation to the standard forms

$$Ax + By + C = 0 \quad \text{or} \quad y = mx + b.$$

For instance, the slopes of the tangents, and so of the normals can be seen more readily.

Likewise $(\cos\alpha)x + (\sin\alpha)y = p$ and, in three-dimensional analysis (where similar arguments apply)

$$(\cos\alpha)x + (\cos\beta)y + (\cos\gamma)z = p,$$

do put the horse before the cart.

F. W. K.

1762. *On Pascal's triangle.*

Pascal's triangle is, of course, well known yet few textbooks appear to suggest it for use with elementary considerations of the Binomial Theorem. Not only do the horizontal lines indicate the numerical coefficients of the terms in the expansion of $(a + b)^n$, for n a positive integer, but also the slant lines give the coefficients in the expansion of $(1 - x)^{-n}$; and this latter fact is completely ignored in every one of a dozen books referred to, although two mention the triangle. Is there some objection to its use, if only as a mnemonic?

F. W. K.

1763. *Lessons on two-dimensional vectors.*

Air navigation has popularised vectors to an extent that few of us dreamed of five years ago. Nowadays, any boy of sixteen whom you pass in the street

is liable to know what a vector is and how to add two vectors together. It therefore seems worth while reconsidering the position of vectors in school mathematics.

Usually it is only those who do mechanics who meet vectors at all, and these may successfully learn about the composition of forces and velocities without ever considering the vector as an entity in itself. No study is made of vectors as such, the possibility of performing other operations than addition is not discussed, and no attempt is made to find other applications or uses.

For beginners, there is a good deal to be said for starting in two dimensions, and it might be thought that on these terms the scalar product would be the utmost limit of their horizon. Prof. Neville, however, has pointed out (Note 1494, *Gazette*, XXIV, p. 357) that in two dimensions the product of the lengths of two vectors and the sine of the angle between them is a scalar quantity. He calls it the *areal product*. It is the natural companion to the ordinary scalar product, and it possesses the interesting property of being non-commutative. It has also two simple applications (areas and moments) and is thus well suited for elementary work.

With these ideas in mind, three lessons were devoted to an experiment in vector work with a top School Certificate " set ". The class had done some statics, including composition of forces, " moments " and " work ". They also knew about sines and cosines of angles of any magnitude, and a little about projections. This last was revised in a preliminary lesson and extended to include the sum of the projections of several vectors. (" Northing " and " Easting " of a ship whose course was several times altered provided a good example ; another was the projection of the sides of a regular hexagon.) In this work, as in the resolution of forces, cosines were used for projecting on one axis and then the sines of the same angles for projection on the perpendicular axis. An axis always bore an arrow at one end and displacements were counted + or − accordingly. Vectors, in fact, and their projections, were already being used, and the theorem that the projection of the sum was equal to the sum of the projections had been made familiar.

In the first lesson on vectors as such, addition was first considered. It was a fairly familiar idea, as applied to forces, and the application to velocity of an aeroplane was either known or easily picked up. The commutative law was seen to apply and was duly greeted with cries of " obvious ". (" Never mind, you'll get a surprise one day.") Subtraction was then considered, rather more briefly. Had more time been available, some geometrical applications would have been worth while at this stage (*e.g.* vector to mid-point of a segment, followed by theorem on lines joining mid-points of opposite sides of a quadrangle).

The question of multiplication was next discussed. It was obvious that the angle must be brought in somehow, and the betting was fairly even as between sine and cosine. Scalar and areal products were then introduced, with the names and the dot and cross notation. (Yellow chalk was used throughout for vectors, white for their lengths.) Applications to mechanics were considered (" work " and " moments ") and the special cases $\theta = 0°$ and $90°$ were related to these. It was then pointed out that as $\cos(-\theta) = \cos\theta$, it did not matter in which direction the angle was taken in evaluating the product $\mathbf{p} \cdot \mathbf{q}$. In fact, $\mathbf{p} \cdot \mathbf{q} = \mathbf{q} \cdot \mathbf{p}$. This was thought very right and proper. The next statement naturally aroused a storm of protest, partially allayed by a further reference to the moment of a force about a point, which was well known to be counted + or − according to the direction of rotation. The lesson was concluded by a short sermon on preconceived ideas.

At the beginning of the second lesson, graph paper was distributed and vectors were drawn from the origin to the points (4, 2) (1, 3). The sum was

found by drawing and it was at once seen that it could be found by adding components. This of course was an example of the theorem that the projection of the sum of two vectors was equal to the sum of the projections.

The scalar product was next found, by measuring the angle and the lengths and using tables. The angle was exactly 45° (a lucky fluke, as the points had been chosen at random), and the scalar product was found to be approximately 10. Could this have been found by combining the original coordinates in some way? To answer this, a simpler case was chosen, using vectors drawn to the points (4, 2) and (2, 0). What was the scalar product? Three boys said " 8 " simultaneously, but on enquiry it appeared that they had all obtained it as $4+2+2$! The right answer was discovered, however, and this led to the discovery that the distributive law could be applied to vectors. This was another " obvious " law of algebra, but a few illustrations of common elementary mistakes in algebra showed that it was at least an important one. It was seen to be equivalent to the theorem about the sum of the projections.

It was now easy to find the scalar product of any two vectors defined by cartesian coordinates. A very few minutes sufficed to do the same for areal products, the only new feature being the minus. (The scalar and areal products of the two vectors first chosen both came to 10, another fluke, not so lucky.) Areas of parallelograms and triangles could now be found, and it was found convenient to arrange the coordinates in a square, like a determinant.

The third lesson began with a revision and summary of results. This was done using vectors $\mathbf{r}_1$, $\mathbf{r}_2$, inclined at θ_1, θ_2 to the axis of x. The corresponding components were (x_1, y_1) and (x_2, y_2). It was found that in this notation

$$\mathbf{r}_1 . \mathbf{r}_2 = x_1x_2 + y_1y_2$$

and

$$\mathbf{r}_1 \times \mathbf{r}_2 = x_1y_2 - x_2y_1.$$

It was a short step further to obtain

$$\cos(\theta_2 - \theta_1) = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2$$

and

$$\sin(\theta_2 - \theta_1) = \cos\theta_1 \sin\theta_2 - \sin\theta_1 \cos\theta_2.$$

By way of postscript, a few minutes were devoted the following week to the use of the scalar product for proving the triangle formula

$$a^2 = b^2 + c^2 - 2bc\cos A,$$

which the class had learnt previously by the ordinary methods.

The work covered in these three lessons might well have been spread over six. The experiment is recorded here in the hope that others may explore the possibilities more fully. E. H. LOCKWOOD.

1764. *Inversion of order in the second derivative.*

For a writer somewhat uncertain of his ground to take refuge in vagueness is not an unheard-of delinquency, but it is seldom that really careful writers come to grief in a deliberate attempt to cross the t's and dot the i's of a standard argument. Precisely this fate has befallen more than one mathematician in using the mean value theorem to show that under certain conditions of continuity the second derivatives f''_{xy}, f''_{yx} of a function $f(x, y)$ of two variables are equal.

The proof, well known, is as follows. Write

$$E(x, y) = f(x+h, y+k) - f(x, y+k) - f(x+h, y) + f(x, y),$$

$$F(x, y) = f(x+h, y) - f(x, y), \qquad G(x, y) = f(x, y+k) - f(x, y).$$

Then since $E(x, y) = F(x, y+k) - F(x, y)$, there is a θ_2 between 0 and 1 such that

$$E(x, y) = kF'_y(x, y+\theta_2 k) = k\{f'_y(x+h, y+\theta_2 k) - f'_y(x, y+\theta_2 k)\},$$

and by a second application of the same mean value theorem, there is a θ_1 between 0 and 1 such that

$$E(x, y) = hkf''_{yx}(x+\theta_1 h, y+\theta_2 k).$$

Similarly since $E(x, y) = G(x+h, y) - G(x, y)$ there are θ_3, θ_4 between 0 and 1 such that

$$E(x, y) = hkf''_{xy}(x+\theta_3 h, y+\theta_4 k),$$

and it follows that if the two second derivatives are continuous they are equal.

There is nothing amiss with this proof. But every now and again a well-intentioned writer tries to make the argument clearer by saying that the second application of the mean value theorem is *to the function* $f'_y(x, y+\theta_2 k)$ *of* x. And this, unfortunately, is nonsense. The θ of a mean value theorem is not an absolute constant: it depends on every parameter that occurs in the function; in the most general sense, the θ is a function of any parameters there may be, but we have no grounds for treating it as a differentiable function or even as a continuous one. Thus $f'_y(x, y+\theta_2 k)$ involves x through θ_2 as well as explicitly, and is not a function of x to which a mean value theorem is known to be applicable.

It is to be emphasised that it is not the application of the mean value theorem that is at fault; what is wrong is the assertion that the function to which the theorem is applied is $f'_y(x, y+\theta_2 k)$, a function which for the immediate purpose could be expressed more fully as $f'_y\{x, y+k\theta_2(x)\}$. The difference required is not

$$f'_y\{x+h, y+k\theta_2(x+h)\} - f'_y\{x, y+k\theta_2(x)\},$$

which is not estimable, but

$$f'_y\{x+h, y+k\theta_2(x)\} - f'_y\{x, y+k\theta_2(x)\},$$

which is correctly replaced by $f''_{yx}\{x+\theta_1 h, y+k\theta_2(x)\}$.

Goursat, perhaps alone of the classical writers, expands the argument and makes no mistake. He says simply that the theorem is applied *to the function* $f'_y(u, x+\theta_2 k)$ *of* u. But it is to be feared that his readers either take the u as a misprint for x or regard the change as a pointless elaboration. Even a Goursat cannot bring about a reform tacitly by example alone. E. H. N.

1765. *Rough pegs, and a problem in algebra hanging on them.*

Three equally rough pegs A, B, C of the same circular cross-section are placed at the corners of an equilateral triangle of side a, so that BC is horizontal and A above BC. Shew that the greatest weight which can be supported by a weight W tied to the end of a string, which is carried once round the pegs and never completely surrounds any peg, is $We^{5\mu\pi}$, μ being the coefficient of friction.

This question, which is to be found in more than one textbook, is a perfect exercise in applied mathematics—all the difficulties are in elementary algebra and trigonometry.

The problem is to wind the string in such a way as to secure the greatest total length of contact with the pegs. It is quite easy to arrange contacts aggregating 10 right angles, but most students shirk the effort of proving

that no improvement on this is possible; if we start with no assumptions as to where the ends are hanging or whether the tangential portions of the string are direct or transverse, the number of cases to dispose of is terrifying; if we draw diagrams, it is hard to know how much argument to indulge in. The crucial step in the following demonstration is not in any sense profound, but it is unusual enough to be worth recording.

Either the string coils in the same direction round all three pegs, or it winds in one direction round two of them and in the other direction round the third. Hence with the proper choice of direction of angular measurement, the total change of direction is expressible either as $(a+b+c)\pi$ or as $(a+b-c)\pi$, where a, b, c are positive and the order of the symbols is not related to the order of the pegs along the string. Since the direction of the string is in fact reversed, $a+b+c$ in the one case or $a+b-c$ in the other case is an odd whole number N, which is positive in the first case but may be either positive or negative in the second case. Since no peg is completely surrounded, each of the angular factors a, b, c is less than 2. All that we have to show is that, with these conditions, $a+b+c \leqslant 5$.

In the first case, $N=a+b+c<6$, and therefore, since N is an odd whole number, $N \leqslant 5$. In the second case, $a+b=c+N$, whence

$$a+b+c=2c+N<4+N, \quad a+b+c=2(a+b)-N<8-N;$$

but either $N \leqslant 1$, algebraically, and the first inequality implies $a+b+c<5$, or $N \geqslant 3$, and the same conclusion follows from the second inequality. Thus without referring to the geometrical configuration, we have $a+b+c \leqslant 5$ always, with equality impossible unless the string winds in the same direction round all the pegs.

It will be observed that in the question as set, most of the data are irrelevant. The sections of the pegs may be any three oval curves, and the geometrical condition under which the frictional factor $e^{5\mu\pi}$ is attainable, though not easy to state briefly, has nothing to do with symmetry or equality.

The argument can be adapted to an arbitrary number p of pegs. We have to prove that if $a_1, a_2, \ldots, a_m, c_1, c_2, \ldots, c_n$ are positive numbers, where $m+n=p$, if each of these p numbers is less than 2, and if the difference $\Sigma a - \Sigma c$ is an odd whole number, then the sum $\Sigma a + \Sigma c$ is less than $2p-1$ unless m or n is zero, in which case the maximum $2p-1$ can be attained.

As before, if $n=0$ the conditions imply $\Sigma a < 2m$, that is, $\Sigma a < 2p$, and therefore $\Sigma a \leqslant 2p-1$; the limit is reached by the simple choice of $2-1/p$ for every a.

If $m \neq 0$, $n \neq 0$, we write $\Sigma a = \Sigma c + N$, and we have

$$\Sigma a + \Sigma c = 2\Sigma c + N < 4n + N, \quad \Sigma a + \Sigma c = 2\Sigma a - N < 4m - N.$$

The odd number N is either less than $2(m-n)$ and therefore not greater than $2(m-n)-1$, or greater than $2(m-n)$ and therefore not less than $2(m-n)+1$; in the one case $4n+N<2(m+n)-1$, and in the other, $4m-N<2(m+n)-1$. Thus in each case we have $\Sigma a + \Sigma c < 2p-1$, as required. E. H. N.

1766. *The trigonometric ratios of angles greater than* 90°.

Col. A. H. Bell, in Note 1605, draws attention to what he considers to be an anomaly in the definition of the trigonometric ratios of angles greater than 90°. It must be admitted that it does exist in most textbooks, but it seems to me that the method suggested is artificial and fails to remove the difficulty. I should be interested to know how Colonel Bell proposes to prove that $\cos 2b = 2\cos^2 b - 1$.

In view of the importance of vectors and sine curves, I have used for some considerable time the projection definition. For this purpose I have designed and used, at the Leicester College of Technology and Commerce, a model consisting of an arm 10″ long pivoted by one end at the centre of a circle of 10″ radius marked in degrees ; at the end of the arm is a rod freely pivoted and having two unequal portions so that it sets itself in a vertical position.

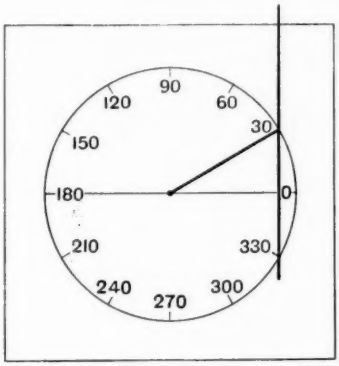


FIG. 1.

Concentrating attention on the height of the tip of the arm, it is easy to show that the sine of the angle grows from 0 to 1 and then decreases to zero. After the 180° mark is passed it is obvious that the sine becomes negative as the tip of the arm is now below the centre. The cosine variation comes later, and is seen in a similar manner by watching the horizontal distance. The tangent can then be derived from these two, and it seems wisest at the start to deal with one ratio at a time. Marking the positive lines red and the negative lines blue helps to emphasise the sign.

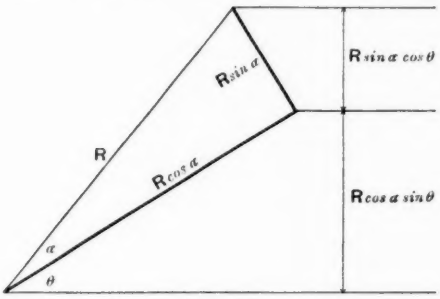


FIG. 2.

This line of approach leads to a natural treatment of the $R \sin(\theta + \alpha)$ transformation, from which formulae of the type $\sin(A + B)$ can easily be deduced.

W. HOWARD JOINT.

1767. *Cycles.*

Mr. R. C. Lyness, in Note 1581, asked for a 7-cycle produced from an iterative relation. Although I have not found a simple solution to the problem, the following analysis shows that if irrational coefficients are allowed, the problem can be solved.

Starting with the relation

$$u_{n+1}u_{n-1}+p=-u_n(u_{n+1}+ku_n+u_{n-1}), \quad \text{.....(i)}$$

which is a generalised form of the first example quoted by Mr. Lyness, write $(n-1)$ for n and subtract and it will be found that

$$u_{n+1}+ku_n-ku_{n-1}-u_{n-2}=0. \quad \text{.....(ii)}$$

It follows that if $k=0$, then $u_{n+1}-u_{n-2}=0$, giving R. C. L.'s 3-cycle.

Now using

$$u_{n-r}+ku_{n-r-1}-ku_{n-r-2}-u_{n-r-3}=0,$$

and putting $r=0, 1, 2, \ldots$, we can remove in turn from equation (ii) the terms u_n, u_{n-1}, etc., and obtain

$$u_{n+1}+au_{n-r-1}+bu_{n-r-2}+cu_{n-r-3}=0,$$

where a, b, c are each functions of k. By choosing the value of k we can arrange each time that $a=0$, $b=0$, $c=-1$.

This problem is the same as that of arranging that for an s-cycle

$$1+kx-kx^2-x^3 \text{ should be a factor of } 1-x^s,$$

or $\quad 1+x+x^2+kx \quad$ should be a factor of $1+x+x^2+\ldots+x^{s-1}$.

(*a*) If s is odd, say, $s=2t+1$, then writing $y=x+1/x$, $y_p=x^p+1/x^p$, $y+k+1$ must be a factor of $1+y_1+y_2+\ldots+y_t$.

Thus we have the table :

for a 3-cycle,	$t=1$,	$y+1=0$,	$k=0$;
for a 5-cycle,	$t=2$,	$y^2+y-1=0$,	$k^2+k-1=0$;
for a 7-cycle,	$t=3$,	$y^3+y^2-2y-1=0$,	$k^3+2k^2-k-1=0$;
for a 9-cycle,	$t=4$,	$y^4+y^3-3y^2-2y+1=0$,	$k^4+3k^3-k=0$;

and so on.

(*b*) If s is even, say $s=2t$, then $1+x+x^2+kx$ should be a factor of

$$1+x+x^2+\ldots+x^{2t-1}=(1+x)(1+x^2+x^4+\ldots+x^{2t-2}),$$

and hence $y+k+1$ must be a factor of

$$y_1+y_3+\ldots+y_{t-1} \quad \text{if } t \text{ is even,}$$

or of $\quad 1+y_2+\ldots+y_{t-1} \quad \text{if } t \text{ is odd.}$

Thus

for a 4-cycle,	$t=2$,	$k=-1$;
for a 6-cycle,	$t=3$,	$k=-2$;
for an 8-cycle,	$t=4$,	$k^2+2k-1=0$;

and so on.

If instead of (i) we start with the relation

$$u_{n+1}u_{n-1}+p=u_n(u_{n+1}+ku_n+u_{n-1}),$$

which is a generalised form of the second and third of the examples given, then

$$u_{n+1}+ku_n+ku_{n-1}+u_{n-2}=0.$$

It follows that for an s-cycle, s must be even, $s=2t$, and $y+k-1$ must be a

factor of $y_1+y_2+\ldots+y_{t-1}$ if t is even or of $1+y_2+\ldots+y_{t-1}$ if t is odd. Hence

for a 4-cycle, $t=2$, $k=1$;

for a 6-cycle, $t=3$, $k=0$ or 2;

for an 8-cycle, $t=4$, $k^2-2k-1=0$;

and so on.

W. HOWARD JOINT.

1768. *Irrational numbers.*

Pythagoras proved that $\sqrt{2}$ is irrational. Professor G. H. Hardy * writes: "There is a passage in Plato's *Theaetetus*, discussed at length by Heath in his *History of Greek Mathematics*, about the attempt of Theodorus to generalize Pythagoras's proof. Theodorus, working some 50 years after Pythagoras, proved the irrationality of $\sqrt{N}$ for all values of N (except square values) up to 17 inclusive. Why, ask the historians, did he stop? Why in any case should it have taken mathematicians like the Greeks 50 years to make so obvious an extension? Zeuthen in particular expended a great deal of ingenuity upon this question, but I think that the ingenuity was misplaced, and that the answer is obvious. Theodorus *did not know the fundamental theorem of arithmetic.*" It is perhaps worth while to note that all but one of the square roots discussed by Theodorus and many other square roots (including $\sqrt{18}$, $\sqrt{19}$, $\sqrt{20}=2\sqrt{5}$, ..., $\sqrt{24}=2\sqrt{6}$, $\sqrt{26}$, ..., $\sqrt{32}=4\sqrt{2}$, and square roots of an infinity of primes) can be shown to be irrational with the help of the observation that any product of odd numbers is an odd number and that therefore if the product of two numbers is even, then at least one of them is even. The famous argument of Pythagoras to show that $\sqrt{2}$ is irrational depends on this same trivial particular case of the fundamental theorem.

Suppose, for instance, that $n=19$, $\sqrt{19}=a/b$, where $(a, b)=1$, that is, the greatest common divisor of a and b is unity. Then $19b^2=a^2$, and a and b are both odd, for otherwise both would be even, thus contradicting the assumption that $(a, b)=1$. Let $a=2k+1$, $b=2l+1$. Then we have

$$19(4l^2+4l+1)=4k^2+4k+1,$$

and therefore

$$19(4l^2+4l)=4k^2+4k-18.$$

Dividing both sides by 2, we arrive at the contradiction that an even number is equal to an odd number.

More generally, if $n\equiv 3$, 5 or 7 (mod 8), and $\sqrt{n}=a/b$, where $(a, b)=1$, then $n(4k^2+4k+1)=4l^2+4l+1$, which is impossible since the expression on the right is congruent to 1 (mod 8) whereas the expression on the left is congruent to n (mod 8), that is, congruent to 3, 5 or 7 (mod 8).

If $n=2m$, where m is odd, and $\sqrt{n}=a/b$ where $(a, b)=1$, then $nb^2=a^2$, a is even, say $a=2c$, $mb^2=2c^2$, and therefore b also is even, which is a contradiction.

Since in our discussion we can neglect square factors of n, and therefore factors of the form 2^{2t}, we see that the above method of reasoning shows that $\sqrt{n}$ is an integer or an irrational in all those cases in which n is not of the form $8\lambda+1$. Perhaps the simplest way of proving that $\sqrt{17}$ is irrational is to observe that if $\sqrt{17}=a/b$, with $(a, b)=1$, then $17b^2=a^2$, and hence $2b^2\equiv a^2$ (mod 3), which is, however, not possible unless a and b are both multiples of 3, a contradiction; the reasoning applies to all numbers $n\equiv 2$ (mod 3).

My thanks are due to my father who helped me in the preparation of this note.

V. T. CHARI.

* *Bulletin of the American Math. Soc.*, 35 (1929) 787.

1769. *Feuerbach's theorem.*

Many proofs of Feuerbach have been published in the *Gazette*, and those I have seen do not include the following; it has the merit of being short, possibly the shortest yet, perhaps even the shortest possible.

Let L be a limiting point of the in-circle (I) and the nine-points circle (N). Let X, Y, Z be the points of contact of I with the sides and let A', B', C' be the mid-points of the sides. Also let b, c and a be in descending order of magnitude.

Since $A'X = \frac{1}{2}(b-c)$, etc., we have

$$B'C'.A'X + C'A'.B'Y = A'B'.C'Z.$$

Also since I, N and L are coaxial,

$$A'L/A'X = B'L/B'Y = C'L/C'Z.$$

Hence
$$B'C'.A'L + C'A'.B'L = A'B'.C'L,$$

so that A', B', C', L are concyclic, that is, N and I touch at L.

W. J. HODGETTS.

1770. *A useful lemma and Pascal's theorem.*

If two involutions have a common pair, X, Y, then the mates of two other points also form an involution with X, Y as follows. If (A, a), (B, b) and (X, Y) are in involution, as are (A, a'), (B, b') and (X, Y), then (a, b'), (a', b) and (X, Y) are also in involution. For

$$\begin{aligned}(abXY) &= (ABYX), \text{ from the first involution,}\\ &= (a'b'XY), \text{ from the second involution,}\\ &= (b'a'YX), \text{ which proves the lemma.}\end{aligned}$$

Use of this lemma proves Pascal's theorem very shortly. Let 12, 45 meet at L; 23, 56 at M; 34, 61 at N. Let LM cut the conic at E and F. Then (2, 1), (5, 4) and (E, F) are in involution since their joins concur at L, as are (2, 3), (5, 6) and (E, F) since their joins concur at M. Hence by the lemma, so are (1, 6), (4, 3) and (E, F); that is, N lies on EF. W. J. HODGETTS.

1771. *The Iso triangle.*

The spelling statistics recently published in the *Gazette*, together with a study of some eight hundred odd geometry scripts, have convinced me that our efforts are in vain.

They will not learn how to spell isosceles.

How would it be to give up the unequal struggle and adopt a standard abbreviation? Isos will not do, for it might so easily become Isoc or Isox, but surely they could learn how to spell an iso triangle.

If " with respect to " can become " wo " and hyperbolic cosine be recognised in " ch ", there would seem to be precedents for the iso triangle.

C. O. TUCKEY.

1772. $2\pi r$ *or* $2\pi r^2$.

It is, I think, universal for teachers to condemn the heresy that the area of a circle is $2\pi r$ as more heinous than the other that the area is $2\pi r^2$. It has not, however, always been easy to convince the offender that the first error is worse than the second.

A recent Higher Certificate question may help here, for in working it $2\pi r$ leads to failure, but $2\pi r^2$ *gets the answer right.*

The question is:

" $ABCD$ is a square of side 1 inch. Circles are drawn with centre A and

radius AB and with centre B and radius BD. Show that the part of the smaller circle which is outside the larger is exactly 1 square inch and that the part of the larger circle which is outside the smaller is greater than the part inside it by exactly 2 square inches." C. O. TUCKEY.

1773. *The construction of logarithm tables.*

I was greatly interested in Mr. Inman's article in the *Gazette* for May 1943. As he states, most of us are entirely unaware of the work that has been done by others in this direction; but I am not sure that this is altogether to be deplored. Original research is an excellent thing for mental development; and from this point of view, it is immaterial whether the field of exploration is entirely new or actually well trodden, provided that it is new to oneself.

I should like to point out, however, although it may be what many others have previously noted, that Mr. Inman's Table F (p. 73), for the computation to ten places of decimals of the logarithm of any number, may be considerably improved in utility by the addition thereto of the numbers whose logarithms are 0·875, 0·75 and 0·625. These numbers are palpably the fourth root of $1000\sqrt{10}$ (or $\sqrt[4]{3162{\cdot}27\ldots}$), the square root of $10\sqrt{10}$ (or $\sqrt{31{\cdot}62\ldots}$), and the fourth root of $100\sqrt{10}$ (or $\sqrt[4]{316{\cdot}2\ldots}$), and therefore may be readily calculated to as many figures as may be desired.

When we require the logarithm of a large number, the use of one of these as the first divisor may reduce the number of terms needed to give the logarithm to the required number of decimal places. EDWARD V. CLARK.

1774. *A vulgar story.*

This is a tale about COMMON LOGS, otherwise known as Logarithms to the base of ten, characterised by $\log 1 = 0$, $\log 10 = 1$, $\log 100 = 2$, $\log 1000 = 3$, and so on.

Being of an enquiring turn of mind, I tried to find out how Log Tables were computed, and sought the information from my mathematical friends.

The replies I received were not very helpful, and were to the effect that I was undertaking a monumental task in my old age, and referred me to a modern standard work on the differential calculus, in support of their contention.

This gave the formula

$$d(\log_{10} x) = \frac{\cdot 4343 dx}{x},$$

which required that dx should be infinitesimal, in which case it certainly would have been a long-winded business.

As logarithms can be set out as a curve, it occurred to me that this curve possessed features in common with the arc of a circle, in that:

(1) $\sin\theta = \theta$, provided the arc is small enough; as an experiment I tried the angle 54′ or one per cent. of 90°.

(2) The chord of an angle, being equal to twice the sine of half the angle, would enable me to use an angle of two per cent.; embodying this into the equation, which now took the form

$$d(\log_{10} x) = \frac{\cdot 4343 dx}{x + \frac{1}{2}dx},$$

with dx not greater than two per cent. of x, ensured correct five-figure logs.

This formula was used to build up the logs of the integral numbers from 1 to 21, the logs so formed were then used to build up log tables by using the multiples, submultiples, powers and roots of the logs themselves, as indicated.

Let $x = 1000$ and $dx = 20$ (2% of x). Apply this to the equation

$$\log(x \pm dx) = \log x \pm \frac{\cdot 4343 \times dx}{x \pm \frac{1}{2}dx},$$

then $$\mathrm{L}(1000+20) = 3 + \frac{\cdot 4343 \times 20}{1000+10},$$

i.e. $$\mathrm{L}\,1020 = 3{\cdot}00860.$$

Similarly $$\mathrm{L}(1020+4) = 3{\cdot}00860 + \frac{\cdot 4343 \times 4}{1020+2},$$

i.e. $$\mathrm{L}\,1024 = \mathrm{L}\,2^{10} = 3{\cdot}01030;$$

hence

$$\begin{aligned}
\mathrm{L}\ 2 &= \tfrac{1}{10}(3{\cdot}01030) &&= \cdot 30103,\\
\mathrm{L}\,5 = \mathrm{L}\,10 - \mathrm{L}\ 2 &= \text{A./C. } \cdot 30103 &&= \cdot 69897,\\
\mathrm{L}\ 4 &= \mathrm{L}\,2 \times 2 &&= \cdot 60206,\\
\mathrm{L}\ 8 &= \mathrm{L}\,2 \times 3 &&= \cdot 90309,\\
\mathrm{L}\ 80 &= \mathrm{L}\,8 + \mathrm{L}\,10 &&= 1{\cdot}90309;
\end{aligned}$$

then $$\mathrm{L}(80+1) = 1{\cdot}90309 + \frac{\cdot 4343}{80\frac{1}{2}},$$

i.e. $\mathrm{L}\ 81 = \mathrm{L}\,3^4 = 1{\cdot}90849$ to nearest 5th decimal.

$$\begin{aligned}
\mathrm{L}\ 3 &= \tfrac{1}{4}(1{\cdot}90849) &&= \cdot 47712,\\
\mathrm{L}\ 6 &= \mathrm{L}\,3 + \mathrm{L}\,2 &&= \cdot 77815,\\
\mathrm{L}\ 9 &= \mathrm{L}\,3 \times 2 &&= \cdot 95424,\\
\mathrm{L}\ 12 &= \mathrm{L}\,3 + \mathrm{L}\,4 &&= 1{\cdot}07918.
\end{aligned}$$

$\mathrm{L}\,1020 = 3{\cdot}00860,$

$\mathrm{L}\ 102 = 2{\cdot}00860.$

$\mathrm{L}\ 17 = \mathrm{L}\,102 - \mathrm{L}\,6,$

$\mathrm{L}\ 85 = \mathrm{L}\,17 + \mathrm{L}\,5.$

$$\mathrm{L}\ 84 = \mathrm{L}(85-1) = 1{\cdot}92942 - \frac{\cdot 4343}{84\frac{1}{2}} = 1{\cdot}92428,$$

$\mathrm{L}\ 7 = \mathrm{L}\,84 - \mathrm{L}\,12,$

$\mathrm{L}\ 11 = \frac{1}{3}\,\mathrm{L}(120+1),$

$\mathrm{L}\ 13 = \mathrm{L}(51+1) - \mathrm{L}\,4,$

$\mathrm{L}\ 19 = \frac{1}{2}\,\mathrm{L}(360+1),$

$\mathrm{L}\ 21 = \mathrm{L}\,7 + \mathrm{L}\,3.$

Examples.

$$\begin{aligned}
\mathrm{L}\ \cdot 3 = \mathrm{L}\,\tfrac{1}{3} = \text{A./C. } 47712 &&= \cdot 52288.\\
\mathrm{L}\ \pi\ (3{\cdot}1416).\\
\mathrm{L}\ 63 = \mathrm{L}\,9 + \mathrm{L}\,7 &&= 1{\cdot}79934.\\
\mathrm{L}\ 62 = \mathrm{L}(63-1) &&= 1{\cdot}79239.\\
\mathrm{L}\ 3{\cdot}1 = \mathrm{L}\,62 - \mathrm{L}\,20 &&= \cdot 49136,\\
\mathrm{L}\ \pi = \cdot 49136 + \frac{\cdot 4343 \times \cdot 0416}{3{\cdot}1 + \frac{1}{2}(\cdot 0416)} &&= \cdot 49715.
\end{aligned}$$

A./C. refers to the Arithmetical Complement, and is most conveniently arrived at by subtracting each cipher or figure from 9, starting with the left and the last and right-hand figure from 10. ERNEST P. CARMODY.

1775. *On the teaching of logarithms.*

Mr. Inman, in his very interesting paper on this subject, mentioned that he had not seen an attempt at a simple explanation of how tables of

logarithms may be constructed previous to that in Hogben's *Mathematics for the Million*.

I have, for some years, in teaching mathematics to boys in Selective Central and Technical Schools, used Curzon's *The Art of Arithmetic*, Book VIII. I do not know the exact date of publication of this series, but it was in use several years before the appearance of Hogben's book. Curzon gives methods of calculating the logarithms of integers from 1 to 10 which are very similar to those of Hogben. Thus

$2^{10} = 1024 \simeq 10^3$; hence may be found the logarithms of 2, 4, 8 ;

$6^9 = 10077696 \simeq 10^7$; from which log 6, log 3 and log 9 emerge ;

$7^4 = 2401 \simeq 3 \,.\, 8 \,.\, 100$, giving log 7.

Curzon then finds 10^a, for $a = \frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{3}{8}$, and so on, by taking square roots and combining results. The results of all these are then used to draw a graph, which is used for simple calculations until the principle is grasped, when tables are introduced. I am aware, of course, that this does not meet Mr. Inman's objection that only 2-figure accuracy is possible by this means, but I find that boys of fourteen are normally satisfied with this explanation as an adequate answer to their first Why.

My own method of approach, having tried several, is now to make use of the natural curiosity of a boy when he is first given a textbook with tables of logarithms at the end. He wants to know what they are for, and by examination of the tables we see how log 2 and log 3 are related to log 6, log 8, log 9 ; we find log 1·0, log 9·999. As soon as the real utility of the tables dawns, someone always asks " Why? " and then we turn to the simple calculation and graphical work mentioned above.

One further point. I have always found the usual textbook method of determining the characteristic to be confusing. Thus we often see : " The characteristic is *one more* than the number of figures before the decimal point ", and " The characteristic is *one less* than the number of noughts after the decimal point ". At the suggestion of a late headmaster I adopted the following method with success, and I wonder how many other teachers have come across, or have used, the method : the table gives the logarithms of all numbers from 1 to 10. Thus 2·7 has a logarithm 0·4314. The numbers are all in " standard form ". If we require log 27·0, the decimal point is *one* place to the right of the standard form position, and the characteristic of the logarithm is *one*. If we require log 0·27, the point is *one* place to the left of the standard form position and the characteristic is *minus one*.

This, in my experience, removes the confusion, but, of course, the standard form idea must have been taught first. This does not imply that multiplication and division of decimals must be done by the " standard form " method, which is not now fashionable, but the answers to multiplication and division sums can easily be expressed in this way. The fact that large and small numbers are so commonly expressed like this by physicists and astronomers is sufficient justification in any case for introducing the notation quite early.

W. A. Parks.

1776. *A note on logarithms.*

As an addition to Mr. Inman's article in the *Gazette*, May 1943, on the construction of logarithm tables, the following suggestions are made.

Let $x/(2+x) = z$, then

$$\log_e (1+x) = \log_e \left\{ \left(1 + \frac{x}{2+x}\right) \Big/ \left(1 - \frac{x}{2+x}\right) \right\}$$

and $\log_e \{(1+z)/(1-z)\} = 2(z + \frac{1}{3}z^3 + \frac{1}{5}z^5 + \ldots)$,

whereas $6z/(3-z^2) = 2(z + \frac{1}{3}z^3 + \frac{1}{9}z^5 + \ldots)$.

The difference between these two expansions is about $8z^5/45$.

Thus, approximately,

$$\log_{10}(1+x) = 6\mu z/(3-z^2), \qquad \mu = \cdot 43429448,$$

with an error of about $z^5/13$.

For example, to find $\log_{10} 2$ we first obtain, as usual, the logarithm of 1·024. In this case, $z = 3/253$.

$$\begin{aligned} \log_{10}(1\cdot 024) &= \cdot 01029995657. \\ \text{Error} &= \cdot 00000000018. \\ \text{Corrected value} &= \cdot 01029995675. \\ \log_{10} 2 &= \cdot 301029995675. \end{aligned}$$

Had we first obtained the logarithm of $1+\frac{1}{9}$, where $z = \frac{1}{9}$, we should have found $\log_{10} 2 = \cdot 3010300$. R. H. BIRCH.

1777. *On Note* 1741.

If xi be substituted for x in formula 1, then it can be shown that

$$(1+xi)^{1/n} = \{(16n - 3nx^2 - 4x^2) + (2n+1)8xi\}/\{(16n - 3nx^2 + 4x^2) + (2n-1)8xi\}.$$

Let $x = 1$, then

$$(1+i)^{1/n} = \{(13n-4) + (16n+8)i\}/\{(13n+4) + (16n-8)i\},$$

and changing this into the form $A + Bi$, we get

$$\tan(\pi/4n) = 336n/(425n^2 - 80).$$

With this as a basis, by interpolation a better result is obtained, namely :

$$\tan(\pi/4n) = 3780n/(4813n^2 - 1000).$$

This is correct to at least four figures for all values of $n > 2$, and as n increases the approximation becomes closer. The other ratios are

$$\sin(\pi/4n) = 3780n/(4813n^2 + 500),$$

$$\cos(\pi/4n) = (4813n^2 - 1000)/(4813n^2 + 500).$$

For values of n between 1 and 2,

$$\tan(\pi/4n) = 3780n/\{(4813n^2 - 1000) - \tfrac{3}{2}(47/n - 25)\},$$

$$\sin(\pi/4n) = 3780n/\{(4813n^2 + 500) + \tfrac{3}{2}(47/n - 25)\}.$$

Alternatively,

$$\sin(\pi/4n) = 355n/(452n^2 + 46\cdot 47 + 3\cdot 4/n^2), \qquad n > 2,$$

correct to at least six figures.

$$\sin(\pi/4n) = 355n/(452n^2 + 46\cdot 41 + 3\cdot 64/n^2), \qquad n > 1,$$

correct to at least four figures. R. H. BIRCH.

1778. *Ferry puzzles.*

The well-known puzzle of ferrying three soldiers and three cannibals across a stream, never leaving the soldiers on either bank outnumbered by the cannibals, in a ferry-boat carrying only two, makes an amusing introductory exercise in coordinate geometry.

If there are p soldiers and q cannibals on one bank then there are $3-p$ soldiers and $3-q$ cannibals on the other ; since the soldiers may not be outnumbered by the cannibals, we have, simultaneously "$p = 0$ or $p \geqslant q$" and "$3-p = 0$ or $3-p \geqslant 3-q$", *i.e.* both "$p = 0$ or $p \geqslant q$" and "$p = 3$ or $q \geqslant p$".

Hence either $p=0$ or $p=3$ or $p=q$. Taking p, q as Cartesian coordinates, the equations $p=0$, $p=3$, $p=q$, with $0\leqslant p\leqslant 3$, $0\leqslant q\leqslant 3$, represent a figure Z lying on its side, the lines $p=0$ and $p=3$ forming the parallel sides and $p=q$ making the cross join. Any point of this figure, with integral coordinates, represents a possible distribution of soldiers and sailors on a river bank, and will be called a *marked* point. A translation from one marked point (p_1, q_1) to another marked point (p_2, q_2) represents a transit from one bank to the other of p_2-p_1 soldiers and q_2-q_1 cannibals. Since the ferry-boat carries at most two men, a possible transit is represented by a translation, parallel to a coordinate axis, from one marked point to another, not more than two units distant, or by a translation along $p=q$ from one marked point to the next.

A translation which increases a coordinate represents a journey one way across the stream, and one which decreases a coordinate, the return journey. Since every crossing is followed by a return, in traversing the diagram every step which increases a coordinate must be followed by one which makes a decrease.

Starting at the origin (and so watching the ferry from the far bank) two initial steps are open to us, one along $p=q$ to the point (1, 1), the other along $p=0$ to (0, 2). Suppose we take the former; this must be followed by a backwards step, the only one open taking us to (0, 1). From here we must go to (0, 3), then back to (0, 2), forwards to (2, 2,), back to (1, 1) and then in

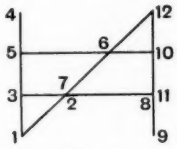


FIG. 1.

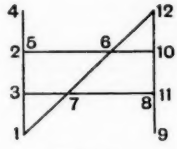


FIG. 2.

turn to (3, 1), (3, 0), (3, 2), (3, 1), coming finally to (3, 3). This solution is shown in diagram 1, the coordinates of each point attained representing the number of soldiers and cannibals on the *far* bank after each crossing. The "forced" character of each step will be immediately apparent from the diagram. For comparison the second solution is shown in diagram 2. The diagrams show there is no solution unless there are two in the boat on at least one return journey.

This method of solution shows at a glance whether the problem of ferrying m soldiers and m cannibals, in a boat carrying n, is solvable or not. For instance, if $m=6$, $n=3$ there is no solution (diagram 3) for we can reach the

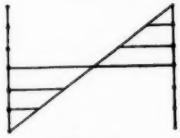


FIG. 3.

line $p=q$ from $p=0$, no higher than the point (3, 3), and since the next step must be backwards, there is no path to $p=3$. The necessary and sufficient condition for a solution is that there is a path from $p=0$ to $p=q$, *above* a path from $p=q$ to $p=3$, and this is satisfied if, and only if, $2n>m$. Thus

with a boat carrying n, the greatest number of soldiers and cannibals that can be taken across the stream is $2n - 1$ of each.

The same diagram (1 or 2) also solves the seemingly more difficult ferry problem in which three ladies and their husbands must be taken across a river in a boat carrying only two, without leaving a lady on either bank in the presence of a man, unaccompanied by her husband. Let the ladies be a, b, c and their husbands A, B, C respectively. Represent a; a and b; a and b and c by $q=1$, $q=2$, $q=3$; and A; A and B; A and B and C by $p=1$, $p=2$, $p=3$ respectively; then the point (3, 2), for instance, represents A, B, C, a, b on one bank. Following diagram 1, we have the solution (indicating the ladies and gentlemen on the far bank after each crossing): Aa; a; abc; ab; $ABab$; Aa; $ABCa$; ABC; $ABCab$; $ABCa$; ABC.

Diagrams 1 and 2 do not give all the solutions of this problem, since certain legitimate combinations are not, nor could be, represented in these diagrams: for instance, a and c together with b and c. Moreover, the solution we have obtained requires that all three ladies and two of their husbands are able to row, whereas it is in fact sufficient if just two of the ladies can row, one of them being unable to row a passenger, and if the husband of the third lady, alone of the men, is able to row. To show this we must extend the diagram. Suppose that A, b and c row, c not taking a passenger.

Represent a; a and b; a and b and c; a and c; c by $q=1$, $q=2$, $q=3$, $q=4$, $q=5$; and A; A and B; A and B and C; A and C; C by $p=1$, $p=2$, $p=3$, $p=4$, $p=5$ respectively. Number each marked point according to the number of persons it represents, then a crossing of the river is represented by a translation from one marked point to another of number not more than two away, parallel to an axis or $p=q$, a translation to a greater number representing an outward journey and one to a lesser number, a return journey (diagram 4).

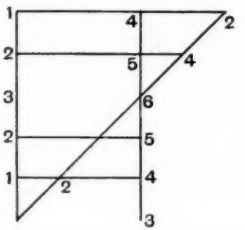


FIG. 4.

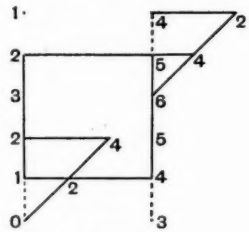


FIG. 5.

If of the possible paths we mark in only those in which A, b, c row, we form diagram 5, and traversing this diagram we obtain the solution (unique apart from an alternative initial move Aa)

ab, a, abc, ac, acAC, cC, ABCc, ABC, ABCab, ABCa, ABCabc.

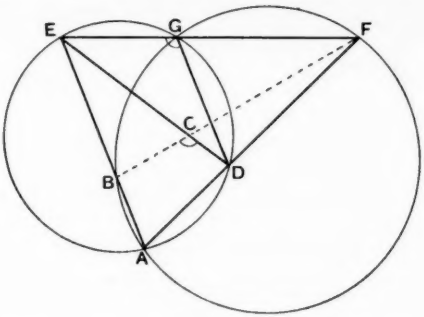
R. L. G.

1779. *A "converse".*

The length of the third diagonal of a cyclic quadrilateral is calculated by means of a geometrical property which is easy to prove, viz. that its square is equal to the sum of the powers of its extremities with respect to the circle; *vide* Hobson's *Trigonometry*.

The converse of this result is not so easy to prove, and it is needed for an important property of inverse points in connection with two orthogonal circles. This converse may be stated as follows: If AEF is a triangle, B a

point on AE and D a point on AF such that $EA \cdot EB + FA \cdot FD = EF^2$, then C, the point of intersection of ED and FB, lies on the circle ABD.



Let the circle EAD cut EF in G. Then $FA \cdot FD = FE \cdot FG$. Hence by subtraction $EA \cdot EB = EF \cdot EG$. Let the circle BAD cut ED in C, and join BC, CF, GD. Then $EC \cdot ED = EB \cdot EA = EG \cdot EF$, by the above. Hence C, D, F, G are concyclic, and so $\angle FCD = \angle FGD$. Also $\angle BCD = 180° - \angle BAD = \angle EGD$. Hence $\angle FCD = 180° - \angle BCD$, so that BCF is a straight line.

Thus if B is the inverse of A with respect to a circle $\mathcal{E}$ centre E, and D the inverse of A with respect to a circle $\mathcal{F}$ centre F which is orthogonal to $\mathcal{E}$, then C is the inverse of B with respect to $\mathcal{F}$, and of D with respect to $\mathcal{E}$.

This in its turn gives the set of images for calculating the distribution of the charge on a conductor (at zero potential) formed by two orthogonal spheres, centres E and F, radii r_1, r_2, in the presence of a point charge e at A. They are $-er_1/AE$ at B, $-er_2/AF$ at D, and

$$(er_1/AE) \cdot (r_2/BF) \quad \text{or} \quad (er_2/AF) \cdot (r_1/DE) \text{ at } C.$$

The equivalence of $AE \cdot BF$ and $AF \cdot DE$ affords a rider on the figure.

N. M. G.

1780. *A close approximation to* π.

We have

$$\frac{1}{3}\left\{\frac{r-\frac{1}{2}}{(r^2-r)(r^2-r+1)} - \frac{r+\frac{1}{2}}{(r^2+r)(r^2+r+1)}\right\} = \frac{1}{r^4} + \frac{1}{r^4(r^6-1)}.$$

Let m, n be two positive integers, $m<n$. Replacing r by $m, m+1, m+2, \ldots n$ in the above identity and adding the results, we obtain

$$\frac{1}{3}\left\{\frac{m-\frac{1}{2}}{(m^2-m)(m^2-m+1)} - \frac{n+\frac{1}{2}}{(n^2+n)(n^2+n+1)}\right\} = \sum_m^n \frac{1}{r^4} + \sum_m^n \frac{1}{r^4(r^6-1)}.$$

Putting $m=2$, and $n=\infty$, we have

$$\frac{1}{12} = \sum_2^\infty \frac{1}{r^4} + \sum_2^\infty \frac{1}{r^4(r^6-1)}.$$

Also

$$\sum_1^\infty \frac{1}{r^4} = \frac{\pi^4}{90}.$$

Hence

$$\frac{\pi^4}{90} = 1 + \frac{1}{12} - \frac{1}{2^4(2^6-1)} - \frac{1}{3^4(3^6-1)} - \frac{1}{4^4(4^6-1)} - \cdots,$$

and $$\pi^4 = 97\tfrac{1}{2} - \left\{\frac{1}{11\cdot 2} + \frac{1}{655\cdot 2} + \frac{1}{11648} + \frac{1}{108500} + \dots\right\}.$$

The series in the bracket appears to have 1/11 as an approximation to its sum ; and in fact

$$\frac{1}{11} - \frac{1}{11\cdot 2} = \frac{1}{616}\,; \quad \frac{1}{616} - \frac{1}{655\cdot 2} = \frac{1}{10296}\,; \quad \frac{1}{10296} - \frac{1}{11648} = \frac{1}{88704} \dots .$$

This proves only that $\pi^4 < 97\frac{9}{22} + \frac{1}{88704}$, say ; but it can be shown that $\pi^4 > 97\frac{9}{22}$ by a very narrow margin. The approximation $\pi^4 \simeq 97\frac{9}{22}$ gives, on taking the square root twice,

$$\pi = 3{\cdot}14159265(3).$$

It remains to show that $$\sum_{2}^{\infty} \frac{1}{r^4(r^6-1)} < \frac{1}{990}.$$

If $f(r) = (r-\frac{1}{2})/9\{(r^2-r)^5 + 5(r^2-r)^4 + 4(r^2-r)^3 + 4(r^2-r)^2 + 2(r^2-r) + 1\}$, it may be shown, by simplification, that $f(r) - f(r+1) - 1/r^4(r^6-1)$ is positive when $r \geqslant 3$, and negative when $r = 2$. Also

$$90f(2) = 1/11\,; \qquad 90f(3) = 1/611{\cdot}08\,;$$

$$90f(4) = 7/72005\,; \qquad 90f(5) = 15/1344547.$$

Let $$\frac{1}{11} - \frac{1}{611{\cdot}08} = \frac{1}{11\cdot 2} - \epsilon_1 \quad \dots\dots(1)$$

$$\frac{1}{611{\cdot}08} - \frac{7}{72005} = \frac{1}{655\cdot 2} + \epsilon_2 \quad \dots\dots(2)$$

$$\frac{7}{72005} - \frac{15}{1344547} = \frac{1}{11648} + \epsilon_3. \quad \dots\dots(3)$$

Adding (1) and (2), $$\frac{1}{11} - \frac{7}{72005} = \frac{1}{11\cdot 2} + \frac{1}{655\cdot 2} + \epsilon_2 - \epsilon_1,$$

so that $$\epsilon_2 - \epsilon_1 = \frac{1}{10296} - \frac{7}{72005} = \frac{7}{72072} - \frac{7}{72005},$$

and this is still negative. But, adding (1), (2), (3),

$$\frac{1}{11} - \frac{15}{1344547} = \frac{1}{11\cdot 2} + \frac{1}{655\cdot 2} + \frac{1}{11648} + \epsilon_3 + \epsilon_2 - \epsilon_1,$$

whence $$\epsilon_3 + \epsilon_2 - \epsilon_1 = \frac{1}{88704} - \frac{1}{89636\frac{7}{15}},$$

which is positive and of order 10^{-7}.

N. M. GIBBINS.

1781. *Application of the Newton approximation to the* m*th root of* N.

Let $f(x) \equiv x^m - N$, $(N > 0)$, so that $f'(x) = mx^{m-1}$; and let a be any positive number.

Then $$a_1 \equiv a - \frac{f(a)}{f'(a)}$$

$$= \frac{1}{m}\left\{(m-1)a + \frac{N}{a^{m-1}}\right\}$$

$$= \frac{a}{m}\left\{m - 1 + \frac{N}{a^m}\right\};$$

and $$a_1^m - N = \left(\frac{a}{m}\right)^m \left\{\left(m - 1 + \frac{N}{a^m}\right)^m - \frac{m^m N}{a^m}\right\}. \quad \text{.....(1)}$$

Let $$\phi(x) \equiv (m - 1 + x)^m - m^m x$$
$$= m^m + m \cdot m^{m-1}(x-1) - m^m x$$
$$+ \binom{m}{2} m^{m-2}(x-1)^2 + \binom{m}{3} m^{m-3}(x-1)^3 + \dots .$$

Then $$\frac{\phi(x)}{(x-1)^2} = \binom{m}{2} m^{m-2} + \binom{m}{3} m^{m-3}(x-1) + \dots ,$$

and $$\frac{\phi(x)}{(1-x)^2} = \binom{m}{2} m^{m-2} - \binom{m}{3} m^{m-3}(1-x) + \dots , \quad \text{.....(2)}$$

If $x > 1$, $\phi(x)/(x-1)^2$ is obviously positive; and if $0 < x < 1$, $\phi(x)/(1-x)^2$ is also positive, the terms in its equivalent being alternately positive and negative and decreasing numerically, since

$$\binom{m}{r+1} m^{m-r+1}(1-x)^{r-1} \div \binom{m}{r} m^{m-r}(1-x)^{r-2} = \frac{m-r}{m(r+1)} \cdot (1-x).$$

Hence $a_1^m - N$ is always positive and $a_1 > N^{1/m}$. By repeated application of (1) we may form a sequence $a_1, a_2, \dots, a_r, a_{r+1}, \dots$; and since

$$a_{r+1} = a_r - f(a_r)/f'(a_r),$$

we have $$a_r - a_{r+1} = (a_r^m - N)/m a_r^{m-1} > 0.$$

Hence $a_{r+1} < a_r$, giving a decreasing sequence.

When $a^m > N$, we have from (2) that

$$a_1^m - N < \left(\frac{a}{m}\right)^m \binom{m}{2} m^{m-2} \left(1 - \frac{N}{a^m}\right)^2$$
$$= \left(\frac{a}{m}\right)^m \cdot \tfrac{1}{2} m(m-1) \cdot m^{m-2} \left(\frac{a^m}{N} - 1\right)^2 \cdot \frac{N^2}{a^{2m}}$$
$$= \frac{N}{2}\left(1 - \frac{1}{m}\right)\left(\frac{a^m}{N} - 1\right)^2 \frac{N}{a^m}$$
$$< \frac{N}{2}\left(1 - \frac{1}{m}\right)\left(\frac{a^m}{N} - 1\right)^2.$$

Hence $$\frac{a_1^m}{N} - 1 < \frac{1}{2}\left(1 - \frac{1}{m}\right)\left(\frac{a^m}{N} - 1\right)^2,$$

and similarly $$\frac{a_r^m}{N} - 1 < \tfrac{1}{2}\left(1 - \frac{1}{m}\right)\left(\frac{a_{r-1}^m}{N} - 1\right)^2. \quad \text{.....(3)}$$

Now $(1 + \epsilon/m)^m > 1 + \epsilon$, so that $(1 + \epsilon)^{1/m} < 1 + (\epsilon/m)$

Hence $$a_r/N^{1/m} < 1 + \frac{1}{2m}\left(1 - \frac{1}{m}\right)\left(\frac{a_{r-1}^m}{N} - 1\right)^2,$$

that is, $$a_r - N^{1/m} < \frac{N^{1/m}}{2m}\left(1 - \frac{1}{m}\right)\left(\frac{a_{r-1}^m}{N} - 1\right)^2$$
$$< \frac{a_{r-1}}{2m}\left(1 - \frac{1}{m}\right)\left(\frac{a_{r-1}^m}{N} - 1\right)^2.$$

Hence, given a_{r-1}, we can, *before calculating* a_r, find out the degree of accuracy to be expected by so doing. For example, if $N=2$, $m=3$, $a_{r-1}=1\cdot26$ we have

$$a_r - 2^{1/3} < \tfrac{1}{6}\,.\,1\cdot26\,.\,(\tfrac{1}{2}\,.\,1\cdot26^3 - 1)^2 = \cdot14\,(\cdot000188)^2$$
$$< 5\,.\,10^{-9}.$$

Also $$a_r = \tfrac{1}{3}(2\cdot52 + 2\,.\,1\cdot26^{-3}) = 1\cdot259921055.$$

Hence $2^{1/3} = 1\cdot25992105$ correct to 8 decimal places.

Again, if $N=6460$, $m=5$, $a_0=6$,

$$a_1 - 6460^{1/5} < \tfrac{6}{5}\cdot\tfrac{1}{5}\cdot\tfrac{4}{5}\left(\tfrac{7778}{6460} - 1\right)^2$$
$$< \cdot48/(4\cdot9)^2$$
$$< \cdot02.$$

Also $$a_1 = \tfrac{1}{5}\left(24 + \tfrac{6460}{1296}\right) = 4\cdot8 + \tfrac{323}{324}$$
$$< 5\cdot8.$$

Hence $$5\cdot8 > 6460^{1/5} > 5\cdot78.$$

Given $a_0 (> N^{1/m})$ we may also estimate the accuracy to be expected from a_r. Successive applications of (3) give

$$\frac{a_r{}^m}{N} - 1 < \left\{\frac{1}{2}\left(1 - \frac{1}{m}\right)\right\}^{2^r-1}\left(\frac{a_0{}^m}{N} - 1\right)^{2^r} \equiv \eta, \text{ say,}$$

so that $$a_r - N^{1/m} < a_0\eta/m.$$

For example, if $N=2$, $m=2$, $a_0=3/2$,

$$a_r - \sqrt{2} < \tfrac{3}{4}\left(\tfrac{1}{4}\right)^{2^r-1}\left(\tfrac{1}{8}\right)^{2^r}$$
$$= 3/32^{2^r}$$
$$= 3/1024^{2^{r-1}}$$
$$< 3/10^3\,.\,2^{r-1}$$

Thus $a_1 = 17/12 < 1\cdot4167$; error less than $2\cdot5/10^3$;

$a_2 = 577/408 < 1\cdot4142157$: error less than $2\cdot2/10^6$.

N. M. GIBBINS.

1782. *Notes on matrices.*

I. If C is a real symmetric matrix, and $x'Cx$ is positive (or non-negative) for all real vectors x, then $\bar{z}'Cz$ is positive (or non-negative) for all complex vectors z.

In fact, if $z = x + iy$, we have

$$\bar{z}'Cz = (x' - iy')C(x + iy) = x'Cx + y'Cy,$$

since $x'Cy = y'Cx$.

This gives an easy proof of the reality of the roots of the equation

$$|\,H - \lambda C\,| = 0,$$

where H and C are Hermitian matrices of the same order and C is positive-definite. In fact, if z is an eigen-vector, we have

$$Hz = \lambda Cz, \quad \text{whence} \quad \bar{z}'Hz = \lambda\bar{z}'Cz.$$

Here everything except λ is real, and the coefficient of λ is non-zero ; hence λ is real.

II. *Rotation in three dimensions.*

A rotation is represented by an orthogonal matrix with positive unit determinant, the rows, and also the columns, being the components of three perpendicular unit vectors in direct order of rotation. Thus if the rows are l_r, m_r, n_r $(r = 1, 2, 3)$, we have $l_1 = m_2n_3 - m_3n_2$, etc. If the angle of rotation is t, the latent roots are 1, e^{it}, e^{-it}. The sum of these being the sum of the elements in the leading diagonal, we have

$$l_1 + m_2 + n_3 = 1 + 2\cos t. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Again, if $a(u, v, w)$ is the unit vector directed along the axis, and T is the matrix, we clearly have $Ta = a$. This gives three equations, equivalent to two, and from these we get, using (1),

$$\frac{2vw}{m_3 + n_2} = \frac{2wu}{n_1 + l_3} = \frac{2uv}{l_2 + m_1} = \frac{u^2}{l_1 - \cos t} = \frac{v^2}{m_2 - \cos t} = \frac{w^2}{n_3 - \cos t} = \frac{1}{1 - \cos t}. \quad \ldots(2)$$

The first two signs of equality give $u : v : w$ linearly. We may note that only the ratios of these are given and only the cosine of t. The reason is that if we reverse the signs both of a and of t we do not alter the rotation. If we change the sign of one or other we get the reverse rotation T'; (we have $T'T = I$).

If we are given the axis and the angle of rotation, we can find the matrix. From (2) we have

$$u(m_3 + n_2) = v(n_1 + l_3) = w(l_2 + m_1).$$

Also, $$m_3^2 + l_2^2 = n_1^2 + n_2^2 (= 1 - n_3^2),$$

whence $$m_3^2 - n_2^2 = n_1^2 - l_3^2,$$

and hence $$\frac{m_3 - n_2}{u} = \frac{n_1 - l_3}{v} = \frac{l_2 - m_1}{w} = 2\theta, \text{ say.}$$

Again, by (2), $$m_3 + n_2 = 2vw(1 - \cos t).$$

We have $m_2 = \cos t + v^2(1 - \cos t)$, and squaring and adding m_1, m_2, m_3, we get $\theta^2 = \sin^2 t$. If we let u, v, w tend to 0, 0, 1 respectively, we see that we must take $\theta = +\sin t$, and we get our matrix in the form:

$$T = \begin{pmatrix} \cos t + u^2(1 - \cos t) & -w\sin t + uv(1 - \cos t) & v\sin t + uw(1 - \cos t) \\ w\sin t + uv(1 - \cos t) & \cos t + v^2(1 - \cos t) & -u\sin t + vw(1 - \cos t) \\ -v\sin t + uw(1 - \cos t) & u\sin t + vw(1 - \cos t) & \cos t + w^2(1 - \cos t) \end{pmatrix}.$$

We can verify this result, or prove it geometrically, by a simple geometrical process. Consider the matrices:

$$A = \begin{pmatrix} u^2 & uv & uw \\ uv & v^2 & vw \\ uw & vw & w^2 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -w & v \\ w & 0 & -u \\ -v & u & 0 \end{pmatrix}.$$

If x is any vector, $\xi = Ax$ is the vector-projection of x on the axis a. Again, $\eta = (I - A)x$ is the projection of x on the plane perpendicular to a. Also $\zeta = Bx$ is the vector-product $a \times x$; it is equal in length to η, and is got by rotating it round a through 90° in the positive sense. The vector x is equal to $\xi + \eta$; the effect of the rotation is to replace the component η by $\eta\cos t + \zeta\sin t$.

That is, $$Tx = \xi + \eta\cos t + \zeta\sin t$$

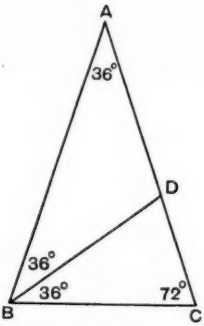
$$= Ax + (I - A)x\cos t + Bx\sin t,$$

or $$T = A(1 - \cos t) + I\cos t + B\sin t,$$

which is the result previously got. **M. F. Egan.**

1783. *Two original proofs.*

(i) To prove that $\cos 36° - \cos 72° = \frac{1}{2}$.



In the figure, $AD = BD = BC = a$, $AC = AB = 2a \cos 36°$.

But $$AC - CD = AD\ ;$$

hence $$2a \cos 36° - 2a \cos 72° = a$$

and $$\cos 36° - \cos 72° = \tfrac{1}{2}.$$

(ii) To find the sum of the series $1^2 + 3^2 + 5^2 + \dots$.

The identities used are

$$\Sigma n^2 = \tfrac{1}{6}n(n+1)(2n+1), \qquad \Sigma n(n+1) = \tfrac{1}{3}n(n+1)(n+2)$$
$$\text{and} \quad (a+b)^2 = a^2 + 2ab + b^2.$$

We have

$$1^2 + 2^2 + 3^2 \quad + \dots + n^2 \quad = \tfrac{1}{6}n(n+1)(2n+1),$$
$$1^2 + 2^2 \quad + \dots + (n-1)^2 \quad + n^2 \quad = \tfrac{1}{6}n(n+1)(2n+1),$$
$$2\,.\,2\,.\,1 + 2\,.\,3\,.\,2 + \dots + 2n(n-1) + 2n(n+1) = \tfrac{2}{3}n(n+1)(n+2).$$

Adding,

$$1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 + 3n^2 + 2n = \tfrac{1}{3}n(n+1)(4n+5),$$

and hence

$$1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = \tfrac{1}{3}n(n+1)(4n+5) - 3n^2 - 2n$$
$$= \tfrac{1}{3}n(2n-1)(2n+1).$$

Cadet A. W. NUTBOURNE, R.N.C., Dartmouth (age 15 yrs. 4 mths.).
[Per D. F. FERGUSON.]

1784. *The logarithmic and exponential properties as particular cases of general relations.*

1. Let $L_n(x) \equiv \int_1^x x^n\,dx$.

(*a*) $$\int_1^h x^n\,dx = \int_k^{hk} \frac{x^n}{k^n} \cdot \frac{dx}{k} = \frac{1}{k^{n+1}} \int_k^{hk} x^n\,dx.$$

Hence $$L_n(kh) = \int_1^{hk} x^n\,dx = \int_1^k x^n\,dx + \int_k^{hk} x^n\,dx\ ;$$

that is, $$L_n(kh) = L_n(k) + k^{n+1} L_n(h),$$

which reduces to the fundamental property of the logarithm when $n = -1$,

or $$k^{n+1} = \frac{(n+1)(k^{n+1}-1)}{(n+1)} + 1 = (n+1)L_n(k) + 1, \quad \text{for } n \neq -1.$$

Hence $$L_n(kh) = L_n(k) + \{(n+1)L_n(k) + 1\}L_n(h)$$
$$= L_n(k) + L_n(h) + (n+1)L_n(k)L_n(h),$$

which tends to the logarithmic property as $n \to -1$.

(*b*) $$L_n(1+x) = \int_1^{1+x} x^n\,dx = \{(1+x)^{n+1} - 1\}/(n+1)$$
$$= \frac{1}{n+1}\left\{(n+1)x + \frac{(n+1)n}{2} \cdot x^2 + \frac{(n+1)n(n-1)}{2\,.\,3}x^3 + \ldots\right\}$$

when $|x| < 1$ (except for special values of n).

Thus, $$L_n(1+x) = x + \frac{n}{2}x^2 + \frac{n(n-1)}{2\,.\,3}x^3 + \frac{n(n-1)(n-2)}{2\,.\,3\,.\,4}x^4 + \ldots, \quad \text{for } n \neq -1,$$

which tends to the series for $\log(1+x)$ as $n \to -1$.

2. Let $E_n(x)$ be the inverse of $L_n(x)$, that is,

$$y = E_n(x) \text{ when } x = L_n(y). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

(*a*) $$x = \int_1^y y^n\,dy = (y^{n+1} - 1)/(n+1),$$

and hence $$E_n(x) = y = \{1 + (n+1)x\}^{1/n+1},$$

which has the well-known limit $\exp x$ when $n \to -1$.

(*b*) By the definition of L_n,

$$dL_n(y)/dy = y^n;$$

so by (i), $$dx/dE_n(x) = \{E_n(x)\}^n,$$

or $$E_n'(x) = \{E_n(x)\}^{-n},$$

which reduces to the fundamental exponential property when $n = -1$.

(*c*) $$E_n(x_1) \,.\, E_n(x_2) = y_1y_2 = E_n\{L_n(y_1y_2)\}$$
$$= E_n\{L_n(y_1) + L_n(y_2) + (n+1)L_n(y_1)L_n(y_2)\}, \quad \text{by I } (a)$$
$$= E_n\{x_1 + x_2 + (n+1)x_1x_2\}, \quad \text{by (i)},$$

which tends to the index property of the exponential function as $n \to -1$.

C. R. Spooner. [Per H. Berry.]

1785. *The invariant of three conics.*

Let $$s_1 \equiv a_1x^2 + b_1y^2 + c_1z^2 + 2f_1yz + 2g_1zx + 2h_1xy,$$

and similarly for s_2 and s_3.

If we write down the condition that $ps_1 + qs_2 + rs_3$ should split into linear factors and equate to zero the coefficient Ψ of pqr, we get a relation between the equations of the conics $s_1 = 0$, $s_2 = 0$, $s_3 = 0$, which is unaltered by change of triangle of reference and homogeneous coordinates (see p. 5 of E. B. Elliott's *Algebra of Quantics*, Clarendon Press, 1913). The invariant Ψ is the sum of the six determinants obtained from

$$\begin{vmatrix} a_i & h_j & g_k \\ h_i & b_j & f_k \\ g_i & f_j & c_k \end{vmatrix}$$

by writing for i, j, k the numbers 1, 2, 3 in any possible order.

Taking $\Psi=0$ and $s_3\equiv z^2$, we find that $s_1=0$ and $s_2=0$ divide $z=0$ harmonically. Hence the well-known tangential equation of the harmonic envelope of $s_1=0$ and $s_2=0$,

$$(b_1c_2+b_2c_1-2f_1f_2)\lambda^2+\ldots+\ldots+2(g_1h_2+g_2h_1-a_1f_2-a_2f_1)\mu\nu+\ldots+\ldots=0,$$

is obtained from $\Psi=0$ by putting $s_3\equiv(\lambda x+\mu y+\nu z)^2$.

Similarly we show that in general $\Psi=0$ is the condition that two of the conics $s_1=0$, $s_2=0$, $s_3=0$ should divide harmonically the chord of contact of the third conic with any of the four conics circumscribing their common self-conjugate triangle and having double contact with the third conic.

Moreover, $s_1=0$, $s_2=0$, $s_3=0$ are the polar conics of points P_1, P_2, P_3 with respect to a cubic curve, which has for its Hessian the Jacobian of the three conics. If $\Psi=0$, P_2 and P_3 are conjugate points for the polar conic of P_1 with respect to this Jacobian (see pp. 237 and 242 of H. Hilton's *Plane Algebraic Curves*, Clarendon Press, 1932).

Again, if $\Psi=0$ and $s_3=0$ is a line-pair, this line-pair is harmonically conjugate with respect to the two lines, passing through their intersection and divided harmonically by $s_1=0$ and $s_2=0$.

If the three conics pass through the same two points, $\Psi=0$ is the condition that they can be projected into circles, such that twice the sum of the squares of their radii is equal to the sum of the squares of the sides of the triangle whose vertices are the centres of the circles.

If the conics touch each other at the same point, $\Psi=0$ is the condition that they can be projected into three parabolas with parallel axes having the sum of the reciprocals of their latera recta zero (with an obvious convention of sign).

If two of the given conics are coincident line-pairs, $\Psi=0$ is the condition that the third conic should pass through their intersection.

We have similar results for four conicoids. Taking

$$s_1\equiv a_1x^2+b_1y^2+c_1z^2+d_1w^2+2f_1yz+2g_1zx+2h_1xy+2l_1xw+2m_1yw+2n_1zw,$$

and so for s_2, s_3, and s_4, we get the invariant Ω, which is the sum of the twenty-four determinants obtained from

$$\begin{vmatrix} a_i & h_j & g_k & l_p \\ h_i & b_j & f_k & m_p \\ g_i & f_j & c_k & n_p \\ l_i & m_j & n_k & d_p \end{vmatrix}$$

by choosing for i, j, k, p the numbers 1, 2, 3, 4 in some order or other.

Putting $s_4\equiv(\lambda x+\mu y+\nu z+\varpi w)^2$ in $\Omega=0$ we get the tangential equation of the conicoid enveloping all planes which meet $s_1=0$, $s_2=0$, $s_3=0$ in three conics having the geometrical relations implied in $\Psi=0$.

If we take $s_3=0$ and $s_4=0$ as coincident plane-pairs, $\Omega=0$ is the condition that their line of intersection should be divided harmonically by $s_1=0$ and $s_2=0$.

If $s_3=0$, $s_3=0$, $s_4=0$ are coincident plane-pairs, $\Omega=0$ is the condition that their point of intersection should lie on $s_1=0$. HAROLD SIMPSON.

1786. *Some suggestions for school work, derived from ancient Egyptian mathematics.*

These ideas occurred to me while delving into ancient Egyptian mathematics. They are quite simple, and, if adapted, are suitable for secondary pupils of any age. I set them down as they arose in my mind. This will save space, and it may interest older pupils to follow my line of thought.

In the Rhind papyrus the area of a circle is found by reducing the diameter by $\frac{1}{9}$ and squaring the result, *i.e.* $(d - d/9)^2$ in modern terms. In speculating as to how the ancient Egyptian arrived at this rule I recalled that he worked with *unit* fractions only, and that his ruler reflected this system, the successive units of the ruler being marked in halves, thirds, fourths, etc. The ruler was, therefore, well adapted for dividing a line into any number of equal parts. To discover the above rule, did the Egyptian reduce the diameter by various unit fractions in turn, construct a square on the reduced diameter, and compare area of square and circle in each case? (We know that he drew on a squared surface, and that his units of area were based on a square. So he could compare areas by counting squares.) I tried this method, drawing, on $\frac{1}{4}$-inch squared paper, quadrants of radius 10, 9, 8, and squares of sides 9, 8, 7, and comparing areas by counting grid squares. I concluded that the Egyptian could have discovered, without much difficulty, that circle 9 and square 8 were most nearly equal. I checked by calculation : $\pi x^2 - 4(x-1)^2$, for $x = 10, 9, 8$. This led me to find, by solving approximately $\pi d^2 = 4(d - xd)^2$, that $(d - 9d/80)^2$ is a nearer approximation than $(d - d/9)^2$. Further, since $\frac{9}{10}$ is reducible to unit fractions, as $\frac{1}{8} - \frac{1}{80}$ or $\frac{1}{10} + \frac{1}{80}$, it would have been quite possible for an Egyptian to have discovered this nearer approximation by an extension of the empiric method I was testing. But this thought provoked an anti-climax. Was not this empiric method far too modern for an ancient Egyptian? Was it not more likely that his rule came from a sudden inspiration? So I tried another line. By drawing circles on a squared surface the Egyptian would no doubt learn which points of intersection of the grid lines lay on particular circles. He certainly knew, about 5000 years ago, that (3, 4) lies on a circle of radius 5 units. Has the 9-unit circle any special feature? Did drawing it give him a clue? Using Pythagoras, $9^2 - 4^2 = 65$, or 8^2 approx. (the approximation is near enough for ancient Egyptian drawing). So (4, 8) lies on a 9-unit circle. I looked at my circle. (I had been too intent on previous work to examine it before.) The solution flashed immediately into my mind. *A square, and a circle through the $\frac{1}{4}$-points of its sides, have approximately the same area.* Here was the origin of the ancient Egyptian rule! But to establish this as a fact needs further evidence. I must limit myself here to school purposes. Taking side of square = 16, then A (area) = 256. $R^2 = 4^2 + 8^2 = 80$. So $A/R^2 = 3\frac{1}{5}$.

Now certain measurements of the Great Pyramid are held by some authorities to show that the Egyptians knew how to calculate the circumference of a circle from the diameter. There is, however, so far as I know, no direct evidence of this knowledge. So my discovery of the $\frac{1}{4}$-point circle's significance set me wondering whether there was any other-point circle which would give a simple rule for finding the circumference. Taking side of square = 2, and circle of equal perimeter passing through $(x, 1)$, then

$$2\pi\sqrt{1 + x^2} = 8.$$

So $x = \cdot 787$. $x = \frac{3}{4}$ is the nearest approximation which gives $R(= \frac{5}{4})$ exactly measurable in the same units ($\frac{1}{4}$) as x. This circle passes through the $\frac{1}{8}$-points of the sides of the square, and C (perimeter of square or circle) $\div d = 3\frac{1}{5}$, as for A/R^2 above. This suggests a method of approaching $C/d = A/R^2 = \pi$ in school, based on the assumption that a square has approximately the same area as the $\frac{1}{4}$-point circle, and the same perimeter as the $\frac{1}{8}$-point circle. Then $C = 4(d - d/5)$, ($x = \frac{4}{5}$ gives the $\frac{1}{10}$-point circle, but no unit fraction formula for C.)

It will be observed that the $\frac{1}{8}$-point circle is a 5-unit circle, each $\frac{1}{8}$-point being a corner of a 3-4-5 triangle. There is nothing to suggest that the Egyptians connected this circle with a square.

R. S. WILLIAMSON.

1787. *Extension of tables of squares.*

Let the number to be squared be

$$N = a_1 a_2 \ldots a_k b_1 b_2 \ldots b_r,$$

where $a_1, \ldots a_k, b_1, \ldots b_r$ are the digits of the number, and $k \leqslant r$.

Let
$$a_1 a_2 \ldots a_k = A,$$
$$b_1 b_2 \ldots b_r = B;$$

then
$$N = 10^r \,.\, A + B$$

and
$$\begin{aligned} N^2 &= 10^{2r} \,.\, A^2 + 2 \,.\, 10^r \,.\, AB + B^2 \\ &= 10^{2r} \,.\, A^2 + (A^2 + B^2) \,.\, 10^r - (A - B)^2 \,.\, 10^r + B^2 \\ &= \left.\begin{aligned} &10^{2r} \,.\, A^2 \quad + B^2 \\ &+ 10^r \,.\, A^2 \\ &+ 10^r \,.\, B^2. \end{aligned}\right\} \quad - (A - B)^2 \,.\, 10^r \end{aligned}$$

Example. To square 897,069, having a table of squares of three-figure numbers.

$$\begin{aligned} 897^2 \times 10^6 + 69^2 &= 804\ 609\ 004\ 761 \\ 69^2 \times 10^3 &= \quad\ 4\ 761 \\ 897^2 \times 10^3 &= \quad 804\ 609 \\ &\quad\ 805\ 418\ 374\ 761 \\ 828^2 \times 10^3 &= \quad 685\ 584 \\ 897{,}069^2 &= 804\ 732\ 790\ 761 \end{aligned}$$

C. ROBERTSON.

1788. *Concerning Note* 1717.

The explanation of the problem illustrated seems to be one of outlook in the teaching.

(i) The London University Matriculation syllabus does not make specific mention of the Sine Rule (in fact it definitely specifies problems soluble by right-angled triangles), but the question papers sometimes include a problem of the type stated. The textbook writers therefore include a solution of the problem, using only the knowledge of right-angled triangles.

(ii) An engineer is taught trigonometry from a utilitarian point of view, and the applications of the Sine Rule and the Link Method are desirable in the solution of many problems. Consequently the practical man will tackle the problem in this manner as stated in the Note.

(iii) If we are striving merely after speed in doing a particular example, then I would point out that by taking the complements of 20° and 30°, we get

$$x = h \tan 60° \quad \text{and} \quad 60 + x = h \tan 70°.$$

Hence
$$h = \frac{60}{\tan 70° - \tan 60°} \text{ feet},$$

a method which incidentally satisfies the outlook of (i). This method is, I think, the way that a surveyor would tackle the problem given. Whilst the practical man's solution provides the most powerful method in that it can be applied to more general types of problem, we must not despise

the right-angled triangle solution of such problems, since there are many who take elementary trigonometry in the Matriculation Examination who do not intend to follow their mathematical training beyond that stage. To such a person I do not think that the extra rules of the engineer's solution are worth while. To analyse the problem into right-angled triangles is a sufficient illustration of mathematical method and well serves the purpose of such a problem.

R. H. PEACOCK.

1789. *A Higher Certificate syllabus of mathematics for students of physics, chemistry, mathematics.**

One important matter appears to have been overlooked. At what stage will the science student have arrived when he begins his higher certificate course? What contribution will his science have made to his mathematical studies? To what extent will the science master co-operate with his mathematical colleague? The S.M.A. advocate General Science—descriptive (in short type tests), non-experimental, qualitative science! Sir J. J. Thomson described the physics of 1910 to the S.M.A. as an "emasculated physics" (see T. A. A. B., *Math. Gaz.*, May 1943, p. 100). How would he describe the phantom physics of general science? The Mathematical Association would render an immense service to education if it could take Physics under its wing and rescue it from the depths of degradation to which it has sunk as a section of general science. Cannot quantitative science and engineering become part of mathematics with a corresponding increase of time to be devoted to it? Any boy or girl of sixteen who has not sufficient common sense or inclination to read qualitative science for himself, has not been educated but instructed, and will merely continue to follow the bell-wether on general science lines.

Now that there is a tendency to discard purely formal methods and to use mechanical and visual aids, it becomes increasingly evident that logic is not the sole criterion of mathematical reasoning. Two types of mathematical training have arisen—the mathematics of school certificates and the pure mathematics of the universities, and the practical mathematics of the technical colleges. It is usual to prescribe two types of mathematics for students of pure science and of engineering. Would it not be better to devise a common syllabus for both? Similarly, we have theoretical and applied mechanics, electricity and electrical engineering, heat and heat engines, hydrodynamics and hydraulics, properties of matter and strength of materials, and so the duplication proceeds. Is it not desirable to combine into one subject all those subjects whose common basis is mathematics and measurement? This subject or group of subjects might be called Mathematical Science and Experimental Mathematics would form part of it.

It seems to me a matter of the highest importance to ensure that the pupil at the pre-higher-certificate stage should develop a background of ideas and methods which lend themselves to natural expansion during the higher certificate stage. There should be more experimental mathematics of the right kind. I do not mean artificial problems devised for the mere sake of doing experiments, for example, each and every method of determining g, but I do mean practical mensuration, surveying, a first-hand knowledge of errors of observation and degree of accuracy, a knowledge of the inevitable variation of things intended to be alike and the numerical measure of these variations, the tabulation and representation of experimental data, how to use laths and string and common sense in improvising apparatus. The student should

* These comments were written *before* the discussion on this topic at the General Meeting in April, 1944.

learn how to apply his mathematics to practical problems, he should develop a geometrical eye which strips objects of their unessential trappings and reduces a machine to a geometrical figure. By now, calculating machines should be as common as chemical balances and no more expensive. The price depends on the demand, and this would come down enormously if quantities were produced such as are really required for school work.

I think the syllabuses are good, but the questions are too formal and fail to discover whether a student can or has applied his mathematics to practical problems. I think that a suitable type of paper is to be found in the Cambridge Qualifying Examination for the Engineering Tripos. More questions of the type to be found in books on practical mathematics such as Perry, Saxelby, Mann, Rose, Brown, Usherwood and Trimble, etc. are needed.

"Simple questions about arrangements and choice and chance" do not appear to me to be as important or as easy as a simple knowledge of statistics.

Important omissions from the syllabus are :

Graphical calculation (nomography, slide rule, log. paper, etc.).

Treatment of observations (laws, errors, representation).

Solid geometrical drawing combined with a little (very little) three-dimensional geometry.

Simple projective geometry would be useful.

Coordinates other than Cartesian might receive a little attention. C. W. H.

1790. *Random sampling numbers and the schoolmaster.*

Reference is made in a review in this issue of the *Gazette* (p. 223) to work on random sampling numbers. This has perhaps some points of special interest to school teachers that have not otherwise been brought out, and an outline of the topic may therefore be of value. Much of the theory of probability originated in a discussion of games of chance and lotteries, and, in consequence, records have been kept of the results of tossing coins or dice. Thus Weldon tossed 12 dice 26,306 times, data have been published of the readings of some gaming tables (for example, some daily bulletins were published of tables at Monte Carlo) and *The Times* publishes the results of draws for the repayment of chance-drawn shares. Data from whist hands have also been recorded. In certain statistical work it is desirable to make allocations by chance, and it was found an economy of time to have tables of randomly sampled numbers for this purpose, as an alternative to drawing numbers out of a hat each time they were needed, or shuffling and dealing cards. As the data already referred to, and similar ones from teetotum spinning and the like, all showed some bias when analysed carefully, alternative methods of compiling random numbers were desirable. The first table so prepared was that by Tippett (published in 1927 as No. 15 of the *Cambridge Tracts for Computers*) which was compiled by noting the last digit from Census of Population and similar returns. He gave 41,600 digits. Fisher and Yates compiled a table of 16,000 digits (published in 1938 in their *Statistical Tables for Biological, Agricultural and Medical Research*) by choosing from certain digits in pages of 20-figure logarithm tables. In the same year Kendall and Babington Smith published (in the *Journal of the Royal Statistical Society*, Vol. CI, part 1) a table of 5000 obtained from a randomising machine. This was followed in 1939 by 100,000 digits (*Tracts for Computers*, No. 24), where the machine gave a digit, one of the ten, on a sector of a rapidly rotating disc as it was exposed at irregular intervals of time by a neon lamp.

In a modern community, however, there is, it seems, no need to construct a randomising machine, for so many features of sociological life exhibit randomness and can be freed from any personal equation. Thus a set of random numbers serviceable for all ordinary purposes can be constructed by

reading the registration numbers of cars as they pass us in the street, for cars though numbered serially move about the streets in a non-serial fashion, obvious errors, such as those of reading the numbers seen every morning on the way to the station along one's own road when Mr. Smith's car is always standing outside No. 49 being, of course, avoided. It is also necessary to take precautions in view of the effect of the latest registration letter being incompletely exhausted. An alternative randomising machine is supplied by the bus tickets issued to any individual in the course of any period of time. These again are numbered serially, usually with a four-digit number, and again such errors as always going out with one's wife and getting two consecutive numbers, travelling every day with the same conductor and buying a ticket of a value seldom issued, can readily be avoided. Using methods such as these—the numbers can afterwards be re-arranged by any rule, or by the numbers themselves, to give additional randomness—tables such as that of which Table I is the beginning can be obtained. Checks of various

TABLE I.

29733	34159	12679	26786	77756
30266	45268	55955	00905	57604
69400	18786	21988	09955	65934
65089	36270	79934	41296	68860
80806	20584	90694	89391	06322

kinds have been devised, again largely by Kendall: they are hardly appropriate to such a small set as we are including here by way of illustration. Such tables of random sampling numbers can be used by the schoolmaster for the following different purposes amongst others: others will suggest themselves in practice as the need arises.

(1) As "tots". The digits can be used for this purpose as they stand. It is difficult to dictate spontaneously random numbers, as omissions, repeats and runs of various kinds are all likely to occur and obscure psychological causes of diverse natures prevent randomness. "Think of a number" is no good. Digits from our list can be used, individually in columns or rows of any required length, or in blocks of any convenient size and arrangement. A convenient one for secondary schools is to add in columns the five five-digit numbers of any block. Some interesting results on the effects of practice and fatigue can be found if this is done: in actual work we have found that the variance between tots is less than the variance between pupils even of a supposedly homogeneous class and less than between days.

(2) If it is desired that random numbers other than on the scale of ten be used, these may be derived in one of several ways. One, if the scale be less than ten, is to use simply the relevant digits as they occur, or, if more than ten, to use the digits in pairs, *i.e.* less than a hundred, and to take the remainder when divided by the radix, omitting all two-figure numbers above the largest multiple of it. As illustration we give Table IIa, in the scale of 2,

TABLE IIa.

01011	10110	11011	10000	11010
01100	00101	10110	10010	10101
11010	00000	00011	11110	01000
11010	10000	11110	01011	10001
00000	01100	10000	10110	00001

as obtained from Table I. Reading from the top left-hand corner horizontally and working in eights we put, for example, 2 = 010, 9 is too much,

7 = 111, 3 = 011, etc. Table IIB is in the scale of 26, where starting as before

TABLE IIB.

NDLMH	*HRPMM*	*PKFVF*	*NKTKJ*	*EBKZZ*
AEWIZ	*UJFAP*	*RTMXQ*	*VEREC*	*JMNAR*
JKBVU	*ZFQID*	*FEVUZ*	*IQSLT*	*JHQAY*
FICKT	*OMIII*	*BPCKD*	*FLDFK*	*AFCYI*
UUCFY	*JSGCY*	*HKLAP*	*IVBZF*	*FKNLF*

(we use more of the original table than is printed in Table I) we take 00 to 25, 30 to 55, 60 to 85, as *A* to *Z*, so that 29 is too much, 73 is *N*, 33 is *D*, 41 is *L*, etc. Our list of randomly sampled numbers brings home vividly the small chance that monkeys randomly hitting a typewriter will spell out a play of Shakespeare! In preparing these from the randomly sampled digits there is inevitably some waste: if there is ample material the methods we have suggested are, perhaps, the simplest.

(3) A number of parallel questions may be required to be set, say of a "multiple choice" type. The order of the questions, and the order of presenting the alternatives can both be obtained by working through a table such as Table I. Thus, if we have five alternatives, we can put them, by working across from the top left-hand of Table I, in the order 2 (from 2) 4 (remainder from 9), omit next (7 gives remainder 2, already used) 3 (from 3), omit 3 and 4 (already used) 1 (from 1) and finally, even if it were not next in the table, 5, the only outstanding digit of the set (or 0, if it had been decided beforehand to use this).

(4) The method of paragraphs (2) and (3) can be used by the sports organiser to assign competitors to heats and stations, and for other games draws. Members of the Mathematical Association will remember that Mr. Hope-Jones has done this by noting the date on a coin drawn from his (or someone else's) pocket. Our table gives us a quicker way for a larger number. Thus to arrange, say, twenty in four heats of five each (for a final of winners and fastest loser) we take Table I to give us

29, *i.e.* 9, *i.e.* 4th station in 2nd heat;
73, *i.e.* 13, *i.e.* 3rd station in 3rd heat;
33, *i.e.* 13, already used.
41, *i.e.* 1, *i.e.* 1st station in 1st heat.

The choice of a starting point in the table can be made by a blindfold stabbing with a pin.

(5) The use of the tables by the mathematical master for illustrations in the school teaching of the theory of probability, etc., is obvious.

It therefore appears that home-made tables of randomly sampled digits can, with due precautions, be prepared by everyday sociological randomising machines, freed from any bias and personal equation, and can be employed usefully in a number of ways by the school teacher. FRANK SANDON.

1791. *On a functional equation.*

The functional equation

$$f(x+y)=\frac{f(x)+f(y)}{1-f(xy)}$$

was studied by Mr. W. K. Haymann in Note 1732, *Gazette*, vol. 28, pp. 114, 115 (July 1944), and it aroused my interest to such an extent that I felt compelled to make a critical examination of his work. Unfortunately the conclusions which I reached are substantially different from his, and equally

unfortunately I did not succeed in getting into touch with him to discuss the problem before submitting the present Note to the *Gazette* ; a few months later, however, a correspondence which passed between us resulted in our agreeing that his result should be replaced by mine.

The result of Mr. Haymann's investigation was the assertion (with slight alterations in his terminology and notation) that the solution of the functional equation which satisfies the initial conditions $f(0)=0$, $f'(0)=\mu$ is

$$f(x)=\mu\exp(\tfrac{1}{2}\mu x^2)\,.\int_0^x\exp(-\tfrac{1}{2}\mu t^2)\,dt.$$

This assertion is not correct unless $\mu=0$; for, as I shall now prove, *the only solutions of the functional equation which are analytic in a domain containing the origin are the (trivial) solutions* (i) $f(x)=-1$, and (ii) $f(x)=0$.

In the functional equation put $x=y=0$; then, writing $f(0)=a$ for brevity, we have

$$a=\frac{2a}{1-a}.$$

The roots of this quadratic equation in a are $a=-1$ and $a=0$. We consider these roots in turn.

Case I. $a=-1$.

In the functional equation put $y=0$ with x arbitrary ; then

$$f(x)=\frac{f(x)-1}{2}\,;$$

and so

$$f(x)=-1$$

with x arbitrary ; and it is a trivial matter to verify that this value of $f(x)$ actually is a solution of the functional equation.

Case II. $a=0$.

In the functional equation put $y=-x$; then

$$0=f(x)+f(-x),$$

and so $f(x)$ is an odd function of x. In particular $f''(0)=0$ as well as $f(0)=0$.

Now expand both sides of the equation

$$f(x+y)\,.\,\{1-f(xy)\}=f(x)+f(y)$$

in ascending powers of y by Taylor's theorem ; for sufficiently small $|y|$ we have

$$\{f(x)+yf'(x)+\tfrac{1}{2}y^2f''(x)+O(y^3)\}\{1-xyf'(0)+O(y^3)\}$$
$$=f(x)+yf'(0)+O(y^3).$$

Equate coefficients of y and y^2 on both sides of this result ; we get

$$f'(x)-bxf(x)=b,\qquad \tfrac{1}{2}f''(x)-bxf'(x)=0,$$

where b has been written for brevity in place of $f'(0)$. These equations are clearly equivalent to

$$f'(x)=b\{1+xf(x)\},\qquad f''(x)=2bxf'(x).$$

Now either $b=0$ or else $b\neq 0$. The first alternative ($b=0$) gives $f'(x)=0$, so $f(x)$ is constant and therefore equal to $f(0)$, which is zero ; and so we have $f(x)=0$ with x arbitrary ; it is again trivial to verify that this value of $f(x)$ actually is a solution of the functional equation.

On the other hand, if we assume the second alternative ($b\neq 0$) to hold, we first differentiate the formula for $f'(x)$, so that we get

$$f''(x)=b\{xf'(x)+f(x)\}\,;$$

and, since $b \neq 0$, a comparison of this value of $f''(x)$ with the value already obtained shows that

$$f'(x) = f(x)/x.$$

Next, a comparison of this value of $f'(x)$ with the value already obtained shows that

$$f(x) = \frac{bx}{1-bx^2}, \qquad f'(x) = \frac{f(x)}{x} = \frac{b}{1-bx^2}.$$

The differential coefficient of $bx/(1-bx^2)$, however, is not equal to $b/(1-bx^2)$ for arbitrary x unless $b = 0$; and so we have obtained a contradiction. The second alternative is therefore untenable, and so the first alternative ($b = 0$), which, as has already been seen, leads to $f(x) = 0$, is the only possibility remaining open when $a = 0$.

This completes the proof of my theorem that $f(x) = -1$ and $f(x) = 0$ are the only solutions of the functional equation which are analytic in a domain containing the origin.

A few words should be said about the permissibility of equating coefficients of y and y^2. Suppose that $f(z)$ is analytic throughout the interior of the circle $|z| = r$, $r > 0$. Let δ and Δ be any positive numbers such that $0 < \delta < \Delta < r$. Let x have any value which lies in the annulus $\delta \leqslant |x| \leqslant \Delta$. Then the expansions of $f(x+y)$, $1 - f(xy)$ and $f(x) + f(y)$ in ascending powers of y are absolutely convergent when $|y|$ is less than the smallest of $r - \Delta$, $r\delta^{-1}$, r. When y is restricted in this way, it is permissible to equate coefficients.

In conclusion I make two comments. The first is that the flaw in Mr. Haymann's work occurs in the tacit assumption at the outset that the functional equation possesses a solution which satisfies his initial conditions with $\mu \neq 0$. If the equation possessed such a solution, then this solution would certainly satisfy his differential equation (3), and his work would have been valid.

The second is that, for positive μ and large positive x, the function

$$\phi(x) \equiv \mu \exp(\tfrac{1}{2}\mu x^2) \cdot \int_0^x \exp(-\tfrac{1}{2}\mu t^2)\, dt$$

is approximately equal to $\sqrt{(\tfrac{1}{2}\mu\pi)} \exp(\tfrac{1}{2}\mu x^2)$.

Hence, for positive μ and sufficiently large positive x and y, all of the three inequalities

$$\phi(x+y) > 0, \qquad \phi(x) + \phi(y) > 0, \qquad 1 - \phi(xy) < 0$$

are satisfied; and this is clearly incompatible with $\phi(x)$ being a solution of the functional equation.

G. N. WATSON.

1792. *Pascal's theorem again.*

A few days after the publication of Note 1511 in the *Gazette* for May 1941 (vol. 25, p. 109), I received a letter from my friend Mr. A. Robson giving a proof of Pascal's theorem and asking whether it was not analytical. The proof in question is as follows:

Call the vertices of the hexagon inscribed in a conic 1, 2, 3, 4, 5, 6, and let the equation of the chord joining any two vertices, r and s, be written in the abridged form

$$(rs) = 0.$$

Then, for any constant value of k, the equation

$$(12)(34)(56) = k(23)(45)(61)$$

represents a cubic locus which obviously passes through the six vertices of the hexagon and also through the three points (12, 45), (23, 56), (34, 61) at which pairs of opposite sides of the hexagon intersect. By suitable choice of k the cubic can be made to pass through an arbitrary seventh point on the conic, and then, since the cubic and the conic have more than six points of intersection, the cubic must degenerate into the conic and a straight line. Since the points (12, 45), (23. 56), (34, 61) do not lie on the conic when the original six vertices are all distinct, these three points must lie on the straight line ; and this is Pascal's theorem.

This proof, with which I should have been familiar, is to be found in the second edition of Salmon's *Higher Plane Curves* (1873), p. 17, and it has doubtless been published in other textbooks as well. My reply to Mr. Robson was that I certainly regarded the proof as analytical, but that I had two objections to it ; the first, which was somewhat frivolous, was that I had a mid-Victorian dislike of using a cubic curve to prove a property of conics and straight lines ; the second, which was more serious, was the fact that the proof depended on a particular case of the theorem that, if a curve of degree m and a proper curve of degree n $(m > n)$ have more than mn points of intersection, then the former curve must be degenerate and consist of the latter curve together with a curve of degree $m - n$. Anyone who has been brought up (as I was) on the porisms to be found in Wolstenholme's *Mathematical Problems* should be sceptical about the obviousness of the truth of a general theorem of this kind, and should avoid using it except as a last resort.

It is possible, at the sacrifice of some simplicity, to construct a proof of Pascal's theorem somewhat similar to the proof just given without the introduction of a cubic curve. This is done by Salmon in his *Conic Sections* (1869), p. 234. My object in writing this Note, however, is to remark that the degeneracy of the cubic can be put in evidence by using the parametric representation of points on the conic as in my Note 1511. Incidentally this verifies the requisite particular case of the above-mentioned general theorem.

Accordingly, take the conic to be $\xi\zeta = \eta^2$ (my reason for changing the current coordinates from x, y, z to ξ, η, ζ will be evident presently), let the parametric representation of any point on the conic be $(1, t, t^2)$, and let the parameters of the vertices of the hexagon be $t_1, t_2, \ldots, t_6$. The equation of the side 12 is then

$$\xi t_1 t_2 - \eta(t_1 + t_2) + \zeta = 0,$$

and the equation of the cubic (before k is fixed) is

$$\{\xi t_1 t_2 - \eta(t_1 + t_2) + \zeta\}\{\xi t_3 t_4 - \eta(t_3 + t_4) + \zeta\}\{\xi t_5 t_6 - \eta(t_5 + t_6) + \zeta\}$$
$$= k\{\xi t_2 t_3 - \eta(t_2 + t_3) + \zeta\}\{\xi t_4 t_5 - \eta(t_4 + t_5) + \zeta\}\{\xi t_6 t_1 - \eta(t_6 + t_1) + \zeta\}.$$

The condition that this cubic passes through the point on the conic with parameter t_7 is evidently

$$\{(t_1 - t_7)(t_2 - t_7)\}\{(t_3 - t_7)(t_4 - t_7)\}\{(t_5 - t_7)(t_6 - t_7)\}$$
$$= k\{(t_2 - t_7)(t_3 - t_7)\}\{(t_4 - t_7)(t_5 - t_7)\}\{(t_6 - t_7)(t_1 - t_7)\} ;$$

so we must take $k = 1$, and the equation of the cubic which should be degenerate is

$$\{\xi t_1 t_2 - \eta(t_1 + t_2) + \zeta\}\{\xi t_3 t_4 - \eta(t_3 + t_4) + \zeta\}\{\xi t_5 t_6 - \eta(t_5 + t_6) + \zeta\}$$
$$- \{\xi t_2 t_3 - \eta(t_2 + t_3) + \zeta\}\{\xi t_4 t_5 - \eta(t_4 + t_5) + \zeta\}\{\xi t_6 t_1 - \eta(t_6 + t_1) + \zeta\} = 0.$$

Multiply out the left-hand side of this equation ; it is not difficult to see that the coefficients of ξ^3, $\xi^2\eta$, $\eta\zeta^2$, ζ^3 all vanish, and some straightforward but rather tedious algebra shows that the surviving terms are expressible in the form

$$\lambda\xi^2\zeta - \lambda\xi\eta^2 + \mu\xi\eta\zeta + \nu\xi\zeta^2 - \mu\eta^3 - \nu\eta^2\zeta$$

with

$$\lambda = t_1t_2t_3t_4 - t_2t_3t_4t_5 + t_3t_4t_5t_6 - t_4t_5t_6t_1 + t_5t_6t_1t_2 - t_6t_1t_2t_3,$$
$$\mu = -t_1t_2(t_4+t_5) + t_2t_3(t_5+t_6) - t_3t_4(t_6+t_1) + t_4t_5(t_1+t_2) - t_5t_6(t_2+t_3) + t_6t_1(t_3+t_4),$$
$$\nu = t_1t_2 - t_2t_3 + t_3t_4 - t_4t_5 + t_5t_6 - t_6t_1.$$

Consequently the cubic actually does degenerate into the conic $\xi\zeta - \eta^2 = 0$ together with the line $\lambda\xi + \mu\eta + \nu\zeta = 0$.

The equation of the line can immediately be thrown into the determinantal form

$$\begin{vmatrix} \xi & \eta & \eta & \zeta \\ 1 & t_4 & t_1 & t_1t_4 \\ 1 & t_2 & t_5 & t_5t_2 \\ 1 & t_6 & t_3 & t_3t_6 \end{vmatrix} = 0,$$

suggested by the determinant given in Dr. Pedoe's Note 1512.

Now consider the ellipse $b^2x^2 + a^2y^2 = a^2b^2$; let θ be the eccentric angle of any point on it, and let $\tan\frac{1}{2}\theta = t$, with corresponding formulae when suffixes are attached to θ and t. We can now identify the homogeneous coordinates ξ, η, ζ with $1 + x/a$, y/b, $1 - x/a$; and so the equation of the Pascal line of the hexagon 123456 inscribed in the ellipse is

$$\begin{vmatrix} 1+x/a & y/b & y/b & 1-x/a \\ 1 & t_4 & t_1 & t_1t_4 \\ 1 & t_2 & t_5 & t_5t_2 \\ 1 & t_6 & t_3 & t_3t_6 \end{vmatrix} = 0.$$

Simple manipulations of the columns reduce this equation to

$$\begin{vmatrix} x/a & y/b & 1 & 0 \\ 1-t_1t_4 & t_1+t_4 & 1+t_1t_4 & t_1-t_4 \\ 1-t_5t_2 & t_5+t_2 & 1+t_5t_2 & t_5-t_2 \\ 1-t_3t_6 & t_3+t_6 & 1+t_3t_6 & t_3-t_6 \end{vmatrix} = 0,$$

and this is again immediately reducible to

$$\begin{vmatrix} x/a & y/b & 1 & 0 \\ \cos\frac{1}{2}(\theta_1+\theta_4) & \sin\frac{1}{2}(\theta_1+\theta_4) & \cos\frac{1}{2}(\theta_1-\theta_4) & \sin\frac{1}{2}(\theta_1-\theta_4) \\ \cos\frac{1}{2}(\theta_5+\theta_2) & \sin\frac{1}{2}(\theta_5+\theta_2) & \cos\frac{1}{2}(\theta_5-\theta_2) & \sin\frac{1}{2}(\theta_5-\theta_2) \\ \cos\frac{1}{2}(\theta_3+\theta_6) & \sin\frac{1}{2}(\theta_3+\theta_6) & \cos\frac{1}{2}(\theta_3-\theta_6) & \sin\frac{1}{2}(\theta_3-\theta_6) \end{vmatrix} = 0.$$

Recently, when turning over the pages of some old volumes of the *Gazette*, I came upon this form of the equation of the Pascal line of a hexagon inscribed in an ellipse in vol. 2 (1904), p. 100 (published in October 1901); it was proposed as a problem for solution by "D.O.S.", whom I have not identified, but no solution seems to have been published hitherto. "Is there any thing, whereof it may be said, See, this is new? It hath been already of old time, which was before us."

G. N. W.

PLYMOUTH BRANCH.

A GENERAL MEETING of the Branch was held on 15th November, 1944, when report and balance sheet covering the period 1st November, 1940, to 31st October, 1944, were adopted.

It is hoped that, as transport and other difficulties ease, Branch activities will resume their normal course. Meetings have been arranged for January 20th, at Truro, and May 5th, at Exeter; others will be held at Plymouth, and at other centres if requested.

REVIEWS.

(i) **Mathematical Statistics.** By S. S. WILKS. Pp. xi, 284 (litho-printed). 25s. 1943. (Princeton University Press)

(ii) **The advanced theory of Statistics. I.** By M. G. KENDALL. Pp. xii, 457. 42s. 1943. (Chas. Griffin)

These two books reached the reviewer's desk simultaneously, and they both appear to deal with statistics. But there the resemblance ceases.

Wilks has written his book as a text in mathematics with the minimum of excursions into applied mathematical statistics problems. It is based on his two-semester course for advanced undergraduates and beginning graduate students, introductory to the more recent developments in the mathematical theory of statistics, a theory which deals with the drawing of inferences from a sample of a larger population. It incorporates much work not otherwise readily accessible : the bibliography has practically nothing but recent works in it, chiefly British and American, the latter being slightly in the majority. Very few French and German workers are quoted, and hardly any others. The book covers a wide field and a high standard of mathematical technique is required. But even apart from this the book is heavy going. The lithography is confusing to the eyes, particularly in some of the more complicated mathematical formulae and symbolisms, for which this typography is not suited. The verbal expression is not always given in simple form. Thus at the very beginning (p. 2, top) the writer speaks of tossing a coin successively where x is the number of heads in a single toss. Punctuation and, at times, very difficult symbolisms, add to the awkwardness. And the practical statistician will find no applications in examples : there is no pandering to arithmetical taste here. The topics included in the eleven chapters include cumulative distribution functions, probability density functions, moment generating functions (not characteristic functions) with some special distributions. There are two chapters on sampling. Analysis of variance in various forms is the subject of Chapter IX, and there is a good discussion of confidence and tolerance limits and intervals. Among the minor and less commonly treated topics is an interesting section on the theory of runs.

Kendall's book has a very different object : as he says, it is a book on statistics and not on statistical mathematics. He proposes to follow it with a second volume which will deal with certain topics which he lists. But the first volume is an attractive survey of a wide field. There is no talking down to the weakness of the statistician who wishes to avoid a bit of rigorous mathematics, but the needs of the working statistician are kept in mind throughout. The general lay-out of the book is like that followed in the more elementary Yule and Kendall (*Introduction to the theory of statistics*, 1937 and 1940 : not reviewed in the *Gazette*), though the pages are physically larger, being 10″ by $7\frac{1}{4}$″ in place of $8\frac{3}{4}$″ by $5\frac{3}{4}$″. Each chapter has its mathematical treatment, illustrative examples, tables, etc., as they arise, with further references and exercises. In the references the proportion of native writers is larger than in Wilks, British publications being in the great majority, though there are more references than in Wilks to Russian and Italian work. The book is thus more definitely representative of a particular school than is that of Wilks. But it is a very live and growing school, and though each writer employs Stieltjes integrals in the early part of his book, yet Kendall keeps his feet much more on the ground of everyday life.

The chapters run in something like the order that has become fairly accepted for presenting the subject—frequency distributions, measures of location and dispersion, constants (with cumulants—formerly classified with

semi-invariants—moment-generating functions and characteristic functions), some standard distributions, sampling (including something about random sampling numbers *) standard errors and correlations. Between are chapters on probability and likelihood, sampling distributions (including Student's t, Fisher's z and k-statistics and χ^2) association and contingency (using Yule's notation) and the book concludes with a section on rank correlation that is of special interest to those school teachers who face classes of the order of 30 and are in the habit of ranking these pupils. Much of this last, together with the part on quantiles generally, is Kendall's own work. It may be noted, however, that psychological grading depends on an estimate of equality which may lead to two things being assessed as equal that differ by less than the *minimum sensibile* : this does not mean, as Kendall seems to suggest (p. 422), that the ranking is illegitimate, for there may quite well be a real and meaningful measurable quantity or quality behind our attempt to rank. There is an appendix of some tables of the normal distribution, of t, of z, and of χ^2, together with an abac for the latter.

The section on standard error includes some very interesting matter about the estimation by various percentiles and ranges. There is no reference here to the recent development of the quantity control method. Wilks introduces this topic ; perhaps Kendall will deal with this in his next volume, though he does not promise this. We look forward with expectation to this other volume. The present first volume is a great acquisition to the shelves of the working statistician. There is no other volume quite like it known to the reviewer ; it brings together in convenient form a great deal of not readily accessible work ; it is free from serious inaccuracies and errors. And above all it is welded together in such a way as to make of a rapidly growing subject with many ramifications and hazy limitations of subject-matter and scope a coherent body of principles and applications. Statistics developed from several independent sources : the study has united many different problems, and the applications of the science are now extremely wide and varied. Statistical technique is needed in many different fields of study and Kendall's book should prove invaluable to workers in any of them. FRANK SANDON.

Heaviside's Operational Calculus made easy. By T. H. TURNEY. Pp vii, 96. 10s. 6d. 1944. (Chapman and Hall)

The form of title of this book, which the author, in common with several others who have preceded him, has borrowed from Sylvanus Thompson, seems to me to be a misleading one.

If the replacement of mathematics by intricate and long-winded arguments such as have been used here is to be claimed as a simplification in any sense, then it seems necessary to provide evidence of the existence of students to whom it can be so regarded. It is difficult to believe that anyone who is not acquainted with the elements of the classical calculus would ever seriously consider acquainting himself with a detailed knowledge of operational methods of the Laplace-Heaviside type, but assuming for the sake of argument that he (or she) should contemplate such a rash undertaking, a book of this kind would only serve to pile confusion upon confusion.

The book is claimed to be likely to be of inestimable assistance to the electrical engineer desirous of learning his subject without a knowledge of mathematics ; it appears to me that it might conceivably be of some use to the mathematician who is desirous of learning something about electrical engineering.

The author's statement in the preface to the effect that it has nowhere

* I may refer to a Note of mine, " Random Sampling Numbers and the Schoolmaster ", in this *Gazette* (p. 216).

been stated in words that the Laplace transform integrals and the Fourier integrals are mathematically related seems to indicate a remarkably small acquaintance with the standard literature and, in particular, with the work of Davis, Titchmarsh and others.

Great and original as the work of Oliver Heaviside undoubtedly was, it is foolish, and indeed demonstrably false, to assert that it is not capable of *comparatively simple* mathematical justification, but this has not been provided by Dr. Turney. J. H. PEARCE.

Primer of Mechanics and Hydrostatics. By H. A. BAXTER. Pp. viii, 199. 3s. 9d. 1943. (Blackie & Son Limited)

In this textbook, designed to be up to School Certificate standard, " the fundamental entities time and mass and the principle of simultaneous motions, as well as some of the compound ideas arising from them, are shown experimentally." Thus various problems of observation are set for the student, and early sets of examples frequently require discussion answers. It is likely that the average pupil will need more guidance than is provided by the text, not only for the experiments but also for the examples, a number of which could not be tackled by those who had merely learned the pages before. Also, of course, someone will be needed with whom to discuss the problems. But in the hands of a capable teacher the book should be exceedingly useful. All the work necessary for the standard stipulated is covered, the Mechanics leading to " Radius of Inertia ", and the Hydrostatics to "Archimedes' Law " and " Boyle's Law ".

A special word must be given to the examples. Some teachers (there is reason to believe their number may be growing) feel that the questions, especially in a book of this range, are at least as important as anything else. This should encourage inspection of this volume, for the collection of examples is highly recommendable. To frequent groups at appropriate intervals is added a set of 300 miscellaneous questions. Answers are provided.

Finally : The work is excellently printed and bound so that on any standards, and especially on those of war-time, the price is low. F. W. K.

Original tables to 137 decimal places of natural logarithms for factors of the form $1 \pm n \,.\, 10^{-p}$, enhanced by auxiliary tables of logarithms of small integers. By H. S. UHLER. 1942.

Exact values of the first 200 factorials. By H. S. UHLER. 80 cents. 1944. (H. S. Uhler, Yale University, New Haven, Conn.)

So many articles on the calculation of logarithms have appeared in the *Gazette* that, even if paper were plentiful, there would be perhaps little need to say much about the radix or factor methods with which we associate the names of Atwood, Flower, Weddle, Hearne and many others, methods which indeed in essentials go right back to Napier and Briggs. For instance, a very little systematic arithmetic will suffice to derive the expression

$$17 = 2^4 \div \left\{\left(1 - \frac{5}{10^2}\right)\left(1 - \frac{9}{10^3}\right)\left(1 - \frac{2}{10^4}\right)\ldots\right\}.$$

Now for log 17 we turn to Professor Uhler's tables, which supply

$$\log\,(1 - n \,.\, 10^{-p}) \qquad (n = 1, 2, \ldots, 9\,;\; p = 1, 2, \ldots, 69)$$

and

$$\log\,(1 + n \,.\, 10^{-p}) \qquad (n = 1, 2, \ldots, 9\,;\; p = 1, 2, \ldots, 21)$$

to 137 decimal places, with the requisite supplementary tables of $\log 2$, $\log 3, \ldots,$ $\log 10$ and of $\log 10^p$ $(p = 20, 30, \ldots, 90, 110)$. The book is a photolithograph from typescript and is a better example of this useful but sometimes

disappointing process than some other recent productions I have seen. For general mathematical symbolism, the process is not entirely satisfactory, but for digital tables it works quite well. Old-style figures are used; the pages are clear and well spaced.

The second booklet, produced in the same style, considerably extends the range of accessible factorial values. T. A. A. B.

Numeros Heronianos. By B. S. Alfaro. Pp. 53. 1943. (Professor Alfaro, 15, Portsdown Avenue, Golders Green, N.W. 11)

Much material has recently been published in the *Gazette* concerning Apollonian figures, integral cyclic hexagons, and so on; while mathematically trivial, these matters are often entertaining and frequently useful in teaching. Professor Alfaro's little booklet (photographed from typescript) makes an extensive study of Heronian sets, a, b, c, where these are rational numbers such that the Hero number for the area of the triangle of sides a, b, c, namely

$$\sqrt{\{s(s-a)(s-b)(s-c)\}}$$

is also rational. The pace is so easy that no one need fear linguistic difficulties, and a good many interesting elementary results are given. T. A. A. B.

SHEFFIELD BRANCH.

Summary of Hon. Secretary's Report for 1943–4.

The Branch has held five meetings; four of these were well attended.

That on 1st June, 1943, was at Abbeydale Grammar School for Girls. The business—consideration of accounts, election of officers—was transacted with unanimity and despatch. An address on "The Teaching of Geometry in German Secondary Schools up to 1933" was given by Dr. Behrend (of King Edward VII School). Miss Mutley (Headmistress) and staff generously provided a delightful light supper.

The meeting of 21st September, 1943, was held at the University. By general request, Mr. Cox, of the University, gave an illustrated lecture on "The Source of Stellar Energy".

On 9th October, 1943, a meeting was convened at the University to consider the Cambridge Syndicate's alternative syllabus in school geometry; the meeting was fortunate in having the assistance of Mrs. Shuttleworth and Professor Daniell.

Another meeting for the same purpose took place on 23rd October, 1943, at High Storrs Grammar School for Girls. Systematic voting on Mr. Daltry's questionnaire was recorded. Miss Battensby (Headmistress) and staff generously provided a pleasant afternoon tea.

The final meeting was held in the University on 2nd May, 1944. Mr. H. G. Green, of University College, Nottingham, lectured on "The Elements of Plane Algebraic Curves".

Membership of the Branch is about 55; income exceeded expenditure by £2 7s. 10d. The Branch is very grateful to its many friends who have so freely helped in its smooth, pleasant running.

Report of Meeting.

The first meeting of the session 1944–5 took place at High Storrs Grammar School for Girls on 7th October, 1944. Dr. Conn, Dr. Speakman, Mr. Boulton (of the University) dealt with the pre-university mathematics course from the respective standpoints of the physicist, chemist and engineer.

John W. Cowley, *Hon. Sec.*

CORRESPONDENCE.

A FALLACY IN ELIMINATION.

To the Editor of the *Mathematical Gazette*.

SIR,—Can you spare me a little space in which to confess to a fallacy in the Introduction to my recent book? The following argument is used (*Jacobian Elliptic Functions*, p. 44):

If $f(z)$ is any elliptic function, the functions $f(y+z)$, $f(z)$, regarded as functions of z, are on the same lattice, and are therefore connected by an algebraic equation, which can be written

$$\Phi\{f(y+z), f(z)\ ;\ y\}=0. \quad \text{.....(1)}$$

The function Φ is polynomial in $f(y+z)$ and $f(z)$, and involves y parametrically. Interchanging y and z, we have also

$$\Phi\{f(y+z), f(y)\ ;\ z\}=0. \quad \text{.....(2)}$$

I now proceed: "The two equations are identical, since otherwise we could eliminate $f(y+z)$ algebraically and obtain a relation satisfied identically by y and z, contradicting the assumption that y is independent of z."

The inference just is not valid, for the simple reason that in general (1) contains the two functions $f(y)$, $f'(y)$, and (2) contains the two functions $f(z)$, $f'(z)$.

A relation like $\cos^2 x - \cos^2 y = \sin^2 y - \sin^2 x$

may easily be a condition of compatibility and not involve a threat to the independence of the variables. The point can be illustrated from the addition theorem for the sine. Write

$$X=\sin x,\ X'=\cos x,\ Y=\sin y,\ Y'=\cos y,\ Z=\sin(x+y).$$

The addition theorem in the ordinary form is

$$Z=XY'+YX'.$$

The simplest algebraic equation connecting Z and X is

$$Z^2-2Y'XZ+(1-Y^2)X^2=Y^2(1-X^2),$$

that is,
$$Z^2-2XY'Z+(X^2-Y^2)=0. \quad \text{.....(3)}$$

and the corresponding algebraic equation connecting Z and Y is

$$Z^2-2YX'Z-(X^2-Y^2)=0. \quad \text{.....(4)}$$

These two equations are compatible, but not identical.

The two equations linear in Z derived from them in the ordinary way are

$$Z-(XY'+YX')=0,\quad (XY'-YX')Z-(X^2-Y^2)=0.$$

The first of these is the formula from which we began, and the condition of compatibility is the identity

$$X^2Y'^2-Y^2X'^2=X^2-Y^2.$$

If (1) happens to be of the form

$$\Phi\{f(y+z), f(z), f(y)\}=0,$$

instead of
$$\Phi\{f(y+z), f(z)\ ;\ f(y), f'(y)\}=0,$$

the argument is perhaps—I am learning caution—sound. Two distinct equations $\Phi(Z, X, Y)=0$, $\Phi(Z, Y, X)=0$ may of course be compatible, as equations in Z, in virtue of an identity in X and Y—we have only to express

(3), (4) above in terms of $\tan \frac{1}{2}x$ and $\tan \frac{1}{2}y$ to have an example—but because the equations are distinct they can be combined to produce an equation of lower degree in Z which is still rational in X and Y, and the assumption is tacit that (1) is irreducible. It was unfortunate for me that the only application I attempted was to the function $\wp z$, for in this case $\wp' x$ and $\wp' y$ do disappear together and the result is accidentally true. The irreducible equation connecting ns u, ns v, ns w when $u+v+w \equiv 0$ is of the fourth degree in each function and of the eighth degree in all.

E. H. NEVILLE.

Reading.

PS.—I take this opportunity to point out that on p. 38 of the book, g_1, g_3 should read $\frac{1}{4}g_2$, $\frac{1}{4}g_3$ in l. 3, and that on p. 205, three lines above 12·43, K_q should read K_p; on p. 220, line below ·517, the reference should be to ·517, not to ·514; in the last line of p. 240, the relation between the variables should be $u = vv$. There are a number of typographical blemishes, but I have not yet found any other mistakes that could mislead or puzzle the reader.

QUEENSLAND BRANCH.

REPORT FOR THE YEAR 1943-1944.

THE Annual Report was presented to the Annual Meeting of the Branch held on 31st March, 1944. The statement of receipts and expenses for the year was presented and was adopted.

During the year three meetings have been held. The first was the Annual Meeting on 7th May, 1943, at which the President, Professor Simonds, gave an address on "Greek Mathematics". At a General Meeting, held at the University on 6th August, Mr. J. P. McCarthy read a paper on "The signs and symbols of elementary mathematics" and at the second General Meeting, also held at the University on 5th November, Mr. E. W. Jones read a paper on "Harmonic analysis".

The Branch has now a credit balance of £13 10s. 9d. The number of Branch members is 31, of whom 9 are members of the Mathematical Association. Four members of the Branch are on duty with the Forces.

The *Mathematical Gazette* comes to hand regularly and the copies are circulated as usual. The attendance at meetings throughout the year has been satisfactory.

J. P. MCCARTHY, Hon. Sec.

BOOKS RECEIVED FOR REVIEW.

E. H. Neville. *Jacobian elliptic functions.* Pp. xvi, 331. 25s. 1944. (Oxford University Press)

T. H. Turney. *Heaviside's operational calculus made easy.* Pp. vii, 96. 10s. 6d. 1944. (Chapman and Hall)

H. S. Uhler. *Exact values of the first* 200 *factorials.* Pp. 24. 80 cents. 1944. (Yale University)

H. S. Uhler. *Original tables to* 137 *decimal places of natural logarithms for factors of the form* $1 \pm n \cdot 10^{-p}$, *enhanced by auxiliary tables of logarithms of small integers.* 1942. (Yale University)

Tables of Lagrangian interpolation coefficients. Prepared by the Mathematical Tables Project under the sponsorship of the National Bureau of Standards. Pp. xxxvi, 392. 5 dollars. 1944. (Columbia University Press)

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